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THE

ELEMENTS OF GEOMETRY

BY

HENRY W. KEIGWIN

Teacher in the Norwich Free Academy

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1898

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PREFACE.

In this book I have attempted to follow a middle course between the treatise which fully proves the propositions of elementary geometry and the syllabus which contains no proofs whatever. The early propositions are proved at length in order to make clear the form of geometrical demonstration and the details of proof are gradually removed in order to throw the pupil on his own resources. It hardly needs argument to show the wisdom of retaining and intensifying the pupil's interest in the study, and many teachers are convinced that this is best done by expecting easy original work very early and by making the exercises an integral part of the course. How far I have succeeded in giving just the right amount of assistance of course depends on the character of the class. The teacher should be ready to supplement and expand the hints to meet the need of students. This requires judgment, as all teaching does. I shall be especially grateful for criticism and suggestion on this feature of the book.

The order of propositions in Plane Geometry is nearly the conventional order of American text-books, but I have placed in Book I the elementary relations of rectilinear figures and of the circle; in Book II proportional line segments including similar and regular figures; in Book III the relations of areas, and measurement. The constructions are established before they are used in demonstration. This is believed to be pedagogically sound, whatever may be said of the logic of hypothetical constructions; and this arrangement also emphasizes drawing, which is a valuable means of discipline in geometry.

In Solid Geometry success is made to depend on the faithfulness with which the student makes models of his figures as explained in Notes at the end. Experience has shown that most of the puzzling features of Solid Geometry are removed when open models, which can be marked on and rebuilt, are used. A part of the class work should consist in making plane drawings of the models.

My thanks are due to two of my associate teachers, Mr. M. E. Jensen and Miss Charlotte C. Gulliver, for assistance, and I am under special obligation to Miss Elizabeth Keith Price of the Springfield (Mass.) High School for encouragement and valuable suggestions.

H. W. K.

NORWICH, CONN., Sept. 1897.

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GEOMETRY.

PRELIMINARY.

1. Any thing—for example, a book—takes up room. The thing is called in Physical Science *a body.* A body is separated from the space surrounding it by a surface. This surface is a boundary and has no thickness; it is not a thin layer of the body and it is not a thin layer of the surrounding space, but the surface—the common boundary—has no thickness at all.

2. The surface of this page just below is partly black and partly white (Fig. 1). The white part is separated from the black part next it by a line. This line is a boundary and has no breadth. Of course it has no thickness, for the surface has no thickness.

3. Finally, one part of the line may be separated from the next part by a point. This point is a boundary and has no length nor breadth nor thickness.

4. The space which a body occupies is called in Geometry a solid. The boundaries of a solid are surfaces, the boundaries of a surface are lines, and the boundaries of a line are points. But we may regard a surface as existing independently of any particular solid and as simply bounding two adjacent portions of space. Also we may regard a line as existing independently of any particular bounded surface; and a point as simply marking a position and not on any special line.

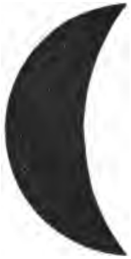

FIG. 1.

The intersection of two surfaces is a line; and the intersection of two lines, or of a line and a surface, is a point.

5. Any combination of points, lines, surfaces, or solids is a figure. Any figure may be moved about in space without any change in size or shape.

6. If a point move from one position to a second position, it will trace out, or describe, a line. If a line move from one position to a second position, it will in general describe a surface. If a surface move from one position to a second position, it will in general describe a solid. We cannot carry this progression further,—there are three steps from point to solid; and for this reason a solid is said to have three dimensions—length, breadth, and thickness,—and space is said to be tridimensional. Similarly, a line has one dimension, and a surface has two dimensions.

7. A plane surface is one "which is of the same shape all over and on both sides"; "that is to say, if we take a body which is partly bounded by a plane surface, we can slide it [the body] all over the surface and it will fit everywhere, and we may also turn it round and apply it to the other side of the surface, and it will fit there too." * A plane surface may be regarded as of indefinite extent.

8. A straight line is one which is of the same shape all along it and on every side; that is to say, if we take a surface partly bounded by a straight line, we can slide that straight boundary all along the straight line, and the two will fit everywhere; and if we turn the bounded surface about the straight line as an axis, the boundary and the line

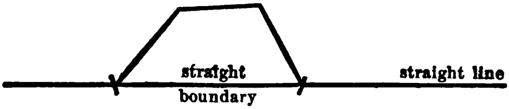

FIG. 2.

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* Clifford.

will continue to fit. A straight line may be regarded as of indefinite length (Fig. 2).

Euclid's definitions of straight line and plane are: A straight line is one that lies evenly as to the points in it; a plane surface is one that lies evenly as to the straight lines in it. These definitions are perhaps nearly equivalent to the explanations given above.

9. The figures which occur in Geometry can be more accurately studied when their definitions are given somewhat formally. Besides the definitions certain other assumptions are made. The Postulates assume that certain figures can be drawn. The Axioms assume that there are certain relations in space.

10. The inferences of Geometry are summed up in the therorems, problems, corollaries, and scholiums.

(1) A **theorem** is a statement which requires proof.

(2) A **problem** proposes a construction which requires solution.

(3) A **corollary** is a theorem (or problem) closely related to a preceding theorem, problem, or definition, and easily deduced from it.

"Proposition" is a general term for a theorem, a problem, or an axiom or postulate.

(4) A **scholium** is a comment upon one or more propositions, showing their relation, extension, etc.

$$\begin{array}{rl} & \left.\begin{array}{l}\text{Definitions}\\ \text{Axioms}\\ \text{Postulates}\end{array}\right\}\text{Assumptions}\\ \text{Propositions}\left\{\begin{array}{l}\text{Axioms}\\ \text{Postulates}\\ \text{Theorems}\\ \text{Problems}\end{array}\right. & \left.\begin{array}{l}\text{Theorems}\\ \text{Problems}\end{array}\right\}\text{Inferences}\end{array}$$

Does the earth's equator have any breadth? or thickness?

Name two boundaries on the earth's surface that intersect in a point.

Is every boundary a "common" boundary (i.e., common to two adjacent magnitudes)?

What is the shape of the upper surface of a cup of still water? That surface is a surface of what else?

If a point is moved, does it *always* trace a line?

If a straight line is moved, does it always trace a surface?

Can a spiral line (such as a "thread" of a screw) be moved so that it shall not trace a surface?

Name any other line which "in general"—but not always—traces a surface.

Name any surfaces that can be so moved as not to describe solids.

Can part of a cone surface be moved and not describe a solid?

Disregarding waves, etc., is the surface of Lake Superior plane?

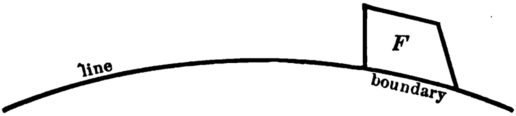


FIG. 3.

In this figure can the boundary of *F* be slid along the line so as to fit everywhere? Does the boundary fit the line "on every side"?

ABBREVIATIONS.

Ax. = Axiom.
Cor. = Corollary.
Def. = Definition.
Post. = Postulate.
Prob. = Problem.
Prop. = Proposition.
Sch. = Scholium.
Theo. = Theorem.

Q. E. D. (*Quod erat demonstrandum*) = which was to be proved.

Q. E. F. (*Quod erat faciendum*) = which was to be done.

SYMBOLS.

∠ = angle.
△ = triangle.
○ = circle.
‖ = parallel.
⊥ = perpendicular.
∴ = therefore.

PLANE GEOMETRY.

BOOK I.

DEFINITIONS. AXIOMS. POSTULATES.

11. A **solid** is a bounded portion of space.

12. A **surface** is a common boundary of two adjacent parts of space. A surface has length and breadth.

13. A **line** is a common boundary of two adjacent parts of a surface. A line has length.

14. A **point** is a common boundary of two adjacent parts of a line. A point has position.

15. A **straight line** is one which lies evenly as to its points. A straight line is often called a right line, or simply a **line.**

16. A plane surface, or a **plane,** is a surface such that, whatever two points in it be taken, the straight line through them lies wholly in the surface.

In this section the plane and the line are supposed to be of indefinite extent; and in general *a line* means a straight line of indefinite length. But a line often means a definite section of line. There should be no uncertainty as to which use of the words is meant.

17. A curved line (or a **curve)** is a line no part of which is straight.

18. A **plane figure** is any combination of points and lines in a plane.

19. An **angle** is a plane figure formed by two straight lines which meet at a point.

The distinction between *lines which meet at a point* (Fig. 4), forming angle x, *one line which meets another* (Fig. 5), and *lines which intersect* (Fig 6), is obvious.

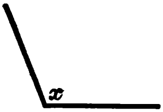


FIG. 4.

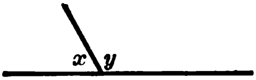


FIG. 5.

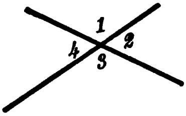


FIG. 6.

The **sides** of an angle are the lines which form the angle.

The **vertex** of an angle is the point where the sides meet.

20. Adjacent angles are those which have a common vertex and a common side between them.

The lettering and reading of diagrams is generally intelligible without explanation. Three points—the vertex and one point on each side—are often used to designate an angle, and the vertex letter is placed between the other two; that is, we read *in* on one side and *out* on the other.

If we suppose a line to revolve like a clock-hand (moved backward) from one position AB to a second position AC, keeping always in the same plane and the point A remaining fixed, the line AB is said to *turn through* an angle BAC. Suppose AB turned in the direction indicated by the arrow. If the revolution is continued to AD, we regard angle BAD as a larger angle than BAC,—in fact angle BAD is the *sum* of angle BAC and angle CAD. The lengths of the sides have nothing to do with the size of an angle.

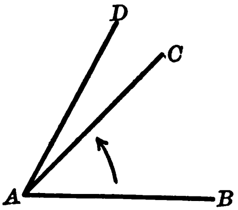


FIG. 7.

When several angles have a common vertex a brief notation is sometimes used, thus: Figures are written within the angles, and the angle so marked is referred to by a subscript figure on the vertex letter. O_1 (read "O one") denotes the angle *AOJ*, O_5 the angle *COL*, etc. One or two other ways of designating angles will be understood when they occur.

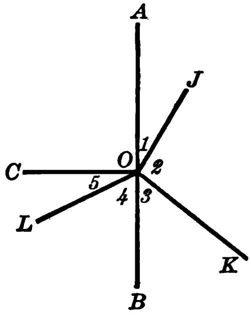


FIG. 8.

When two lines intersect, the opposite angles (such as *a*, *b*) are called ***vertical angles;*** *x* and *y* are also vertical angles.

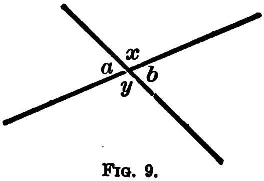


FIG. 9.

Evidently a line, *r*, may be revolved from a first position, *AB*, to a second position, *AC*, by turning it in either of two directions. In other words, when two lines meet there are formed *two* angles.

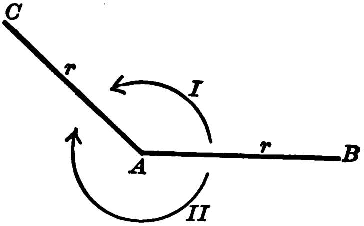


FIG. 10.

21. When two lines meet at a point the two angles formed are called **conjugate angles.** The larger of the two is called the *major*[*] angle, and the smaller one the *minor.*

---

[*] The major angle is sometimes called a reflex angle.

When no distinguishing word or mark is used the smaller one is meant.

22. The assumption in § 5 (repeated in § 40) that figures may be moved about without change in size or shape is useful in *superposing* a figure—i.e., in placing one figure upon another—and also in placing figures adjacent. It should be noticed that when one angle is superposed on another the vertices must coincide and there must be some angle common to the two. And a similar remark may be made of any two figures compared by superposition.

23. The **magnitudes** of plane geometry are lines, angles, and surfaces. Only magnitudes of the same kind can be compared.

24. The **sum** of two magnitudes is the magnitude of the same kind which is formed by placing the two parts adjacent.

Thus the sum of *a* ———————————— and *b* ———— is the line formed by placing *a* and *b* adjacent in the same straight line.

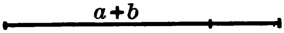


Fig. 11.

The sum of the angles *a* and *b* is the angle *HOL*;

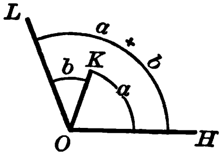


Fig. 12.

and the sum of angles *x* and *y* is the major angle *MOR*.

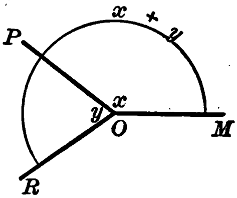


Fig. 13.

25. The **difference** of two magnitudes is the magnitude of the same kind not common when the smaller is superposed on the larger.

Thus $a - b = x$.

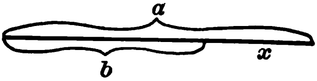


FIG. 14.

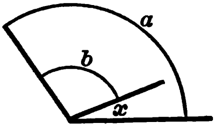


FIG. 15.

26. Two **angles** are **equal** when one can be superposed on the other so that the sides of one fall along the sides of the other (§ 22).

27. Two **figures** are **equal** when one can be superposed on the other so that they shall coincide throughout.

28. **To bisect** any figure is to divide the figure into two equal parts.

29. A **right angle** is either angle formed by one straight line meeting another so that the adjacent angles are equal to each other.

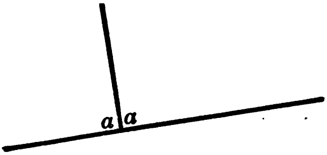


FIG. 16.

30. A **perpendicular** to a line is a line that makes a right angle with it.

31. **Parallel lines** are lines in the same plane which will not meet however far they are produced in either direction.

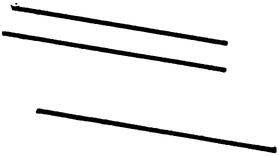


FIG. 17.

32. A **triangle** is a plane figure bounded by three straight lines. The sides of a triangle are the lines which bound it; the vertices are the vertices of its angles.

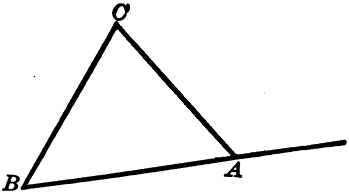


FIG. 18.

33. An **isosceles** triangle is a triangle two of whose sides are equal to each other.

The base of a triangle is any side of the triangle, and *the vertex* is the angular point opposite *the base*; i.e., *base* and *vertex* are relative terms. But when the triangle is isosceles *the base* usually means the side which is not one of the equal sides.

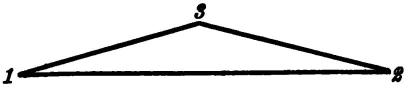


FIG. 19.

A **scalene** triangle has no two sides equal.

34. A **right-angled triangle** (or a **right triangle**) is a triangle one of whose angles is a right angle.

35. AXIOM. *A straight line joining two points is the shortest line which can join the points.*

35a. The **distance between two points** is the length of the shortest line joining those points.

The axiom in § 35 assumes that the shortest line joining two points is a straight line. This axiom may then be stated:

A straight line is the shortest distance between two points.

In this form it is often used as a definition of a straight line.

36. A **circle** is a plane figure bounded by a curved line, every point in the line being equally distant from a point within called the center.

The **circumference** is the bounding line of the circle.

An **arc** is a part of the circumference.

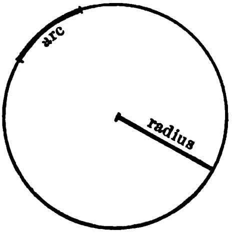


FIG. 20.

37. A **radius** is a straight line from the center to any point on the circumference.

38. COR. *All the radii of a circle are equal.*

EXERCISES. 1. Draw one line to meet another. Letter the figure and read the angles.

2. Draw two lines that intersect; letter and read.

POSTULATES.

39. I. *A straight line can be drawn through any two points.*

II. *A circle can be drawn with any point for center and any length for radius.*

AXIOMS.

40. I. *Through two points there can be only one straight line.*

II. *The straight line joining two points is the shortest line which can join the points.*

III. *Through a given point there can be only one parallel to a given line.*

IV. *Figures may be moved in space without change in shape or size.*

41. NOTES. (1) Postulate I assumes that a straight line or any number of straight lines can be drawn from or through any point; that a straight line can be drawn joining any two points; and that the line joining any two points can be indefinitely extended beyond either or both points.

(2) Postulate II assumes that an arc can be drawn. And Postulates I and II together make it possible to lay off any given length on a line, or to extend a line any given length.

(3) Axiom I assumes that there can be only one straight line joining two points, and that if a straight line is extended beyond a point in the line there cannot be two divergent extensions. This is equivalent to saying:

Ax. I*a*. *Two straight lines cannot intersect twice.*

(4) The above axioms are called *The Axioms* (§ 9) of Geometry. There are other obvious relations true of all quantity, whose statements form the General Axioms of Mathematics. The ones more frequently used in Geometry are collected below for reference. Negative quantity is disregarded in their statement.

EXERCISES. 1. Draw an equilateral triangle (all sides equal).

2. Draw an isosceles triangle on a given base.

GENERAL AXIOMS.

42. (1) Things equal to the same thing are equal to each other.

(2) If the same operation be performed upon equals, the results are equal.

(3) If the same operation be performed upon both members of an inequality, there results an inequality *in the same sense.**

(4) If from the members of an equality there be subtracted the members of an inequality, there results an inequality *in the opposite sense.*

(5) The whole is greater than any of its parts.

(6) The whole is equal to the sum of all its parts.

(7) Of two quantities of the same kind either
the first is greater than the second, or
the first is equal to the second, or
the first is less than the second.

(8) Of three quantities, if the first is greater than the second and the second is greater than the third, then the first is greater than the third.

43. A theorem consists of two parts, the *hypothesis* and the *conclusion.* The hypothesis contains the conditions on which the conclusion depends.

EXERCISES. 1. Draw two isosceles triangles (I and II), each with a given base (AB and HK, equal) and each with a given side (AC and HN, equal).

2. Taking triangles I and II of Ex. 1, superpose I on II, placing AB on HK, with A at H.

---

* The members of an inequality are the quantities which are compared.

Two inequalities are in the same sense when both have the greater quantity first or when both have the greater quantity last.

$$\left.\begin{matrix} a > b \\ x > y \end{matrix}\right\}\text{Same sense.} \qquad \left.\begin{matrix} a > b \\ y < z \end{matrix}\right\}\text{Opposite sense.}$$

TRIANGLES. INTERSECTING LINES.

PROPOSITION I. THEOREM.

44. *If two triangles have two sides and the included angle of the one equal respectively to two sides and the included angle of the other, the two triangles are equal.*

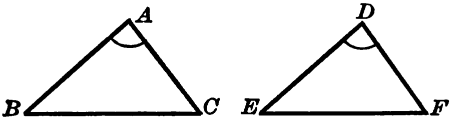


FIG. 21.

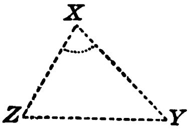


FIG. 22.

Let the two triangles ABC and DEF have the sides AB, AC of the one equal respectively to the sides DE, DF of the other, and let the included angle A be equal to the included angle D.

It is to be proved that the triangles ABC and DEF are equal.

Superpose the triangle ABC upon DEF, applying angle A to angle D, and side AB to side DE.

As AB is equal to DE, [*By hypothesis.*
the point B will fall upon point E. [§ 27. *Two figures are* . . .

As angle A is equal to angle D, [*By hypothesis.*
side AC will fall upon side DF; [§ 26. *Two angles* . . .

and as AC is equal to DF, [*Why?*
point C will fall upon point F. [*Why?*

That is, BC and EF coincide at the points E and F.

Therefore BC and EF coincide throughout. [Ax. I.

That is, triangles ABC and DEF coincide throughout.

Therefore the triangles are equal. [§ 27.

Therefore, *If two triangles have, etc.* Q.E.D.

PROPOSITION II. THEOREM.

45. *If two triangles have two angles and the included side of one equal respectively to two angles and the included side of the other, the two triangles are equal.*

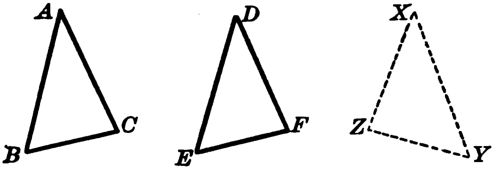


FIG. 28.

Let ABC and DEF be two triangles in which angles B and C of the one are equal respectively to angles E and F of the other, and side BC is equal to EF.

It is to be proved that the triangles ABC and DEF are equal.

Superpose triangle ABC upon triangle DEF, applying side BC to side EF, and angle B to angle E.

As angle B is equal to angle E, [*Why?*
side BA will fall upon side ED, [§ 26.
and point A will fall somewhere on ED or ED produced.

As side BC is equal to side EF, [*Why?*
point C will fall upon point F; ["
and as angle C is equal to angle F, ["
side CA will fall upon side FD, [§ 26.
and point A will fall somewhere on FD.

That is, point A falls upon both ED and FD.

Therefore it must fall at D. [§ 41. Ax. I*a*.

That is, the triangles ABC and DEF coincide throughout.

Therefore they are equal.

Therefore, *If two triangles have, etc.* Q.E.D.

Ex. Prove Prop. II, using *ABC* and *XYZ*. What angles in *ABC* and *XYZ* are *proved* equal?

46. DEF. **Homologous parts** in two figures (as in two triangles) are parts which have the same relative position. When two figures are equal it follows that the homologous parts are equal. The following statement is convenient for reference:

47. COR. *Homologous parts of equal figures are equal.*

NOTE. Observe that the theorem is stated twice: first as a general truth, and then as something true of the particular diagram on the page. The first statement is called the *general enunciation*, and the second is called the *particular enunciation*.

As the diagram pictures *any* figure having the supposed relations, the theorem is proved true in general.

The steps of a proof should be noticed. In Prop. I it is proved (1) that *B* falls on *E*, (2) that *C* falls on *F*; hence (3) the triangles coincide.

EXERCISES. 1. Point out the homologous parts in the triangles of Prop. I. How many pairs of equal homologous parts are used to prove the triangles equal?

2. What are the steps in the proof of Prop. II?

3. In Prop. I superpose triangle *ABC* on *DEF*, making angle *A* coincide with angle *D*, but so that the triangles shall *not* coincide.

4. Draw an isosceles triangle *ABC*, with *AB* and *AC* the equal sides. Lay off *BD* on *BA*, and lay off $CK = BD$ on *CA*. Draw *CD* and *BK*.

(1) Why is $AD = AK$?

(2) What angle in triangle *ABK* is equal to an angle in triangle *ACD*? (Draw the triangles separately.)

(3) In triangles *ABK* and *ACD*, what pairs of sides are equal?

(4) Why is triangle *ABK* equal to triangle *ACD*?

(5) What homologous parts are equal as a result?

Ppoposition III. Theorem.

48. *The angles at the base of an isosceles triangle are equal to each other.*

Let ABC be an isosceles triangle having side AB equal to side AC.

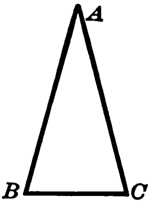


Fig. 24.

It is to be proved that angle ABC is equal to angle ACB.

Extend AB and AC beyond the base; and with A as a center and a radius greater than AB lay off AF equal to AH.

Draw FC, BH. [?

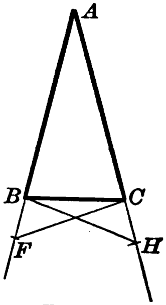


Fig. 25.

Then the two triangles AFC, ABH have

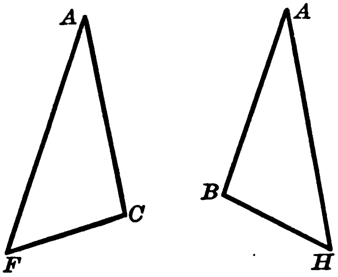


Fig. 26.

$AF = AH$, [?

$AC = AB$, [?

angle A = angle A. [*Common.*

Therefore the triangles AFC, ABH are equal. [§ 44.

Therefore side FC = side BH,

angle AFC = angle AHB,

and angle ACF = angle ABH. [§ 47.

Then subtracting the equals AB, AC from the equals AF, AH, the remainders are equal,

$$BF = CH.$$ [§ 42 (2).

FIG. 27.

Then the two triangles FCB, HCB have

$$FB = HC,$$ [*Just proved.*

$$FC = HB,$$ [" "

$$\text{angle } BFC = \text{angle } CHB.$$ [" "

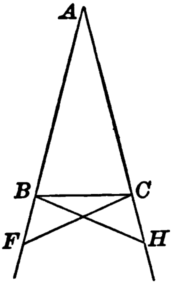


FIG. 28.

Therefore the triangles FCB, HCB are equal. [§ 44.

Therefore angle FCB = angle HBC. [§ 47.

It was just proved that

$$\text{angle } ACF = \text{angle } ABH,$$

and also that

$$\text{angle } FCB = \text{angle } HBC.$$

Therefore, subtracting equals from equals,

$$\text{angle } ACB = \text{angle } ABC.$$ [§ 42 (2).

Therefore, *The angles at the base, etc.* Q.E.D.

49. DEF. A theorem is in **typical form** when the conditions are grouped first in the theorem and all introduced by the word *if*, while the **conclusion** is a single direct statement introduced by the word *then*. Thus, Prop. III stated in typical form is:

"*If a triangle has two sides equal to each other,* ***then the angles opposite those sides are equal to each other.***"

Ex. Prove that an equilateral triangle is also equiangular. State this in typical form.

PROPOSITION IV. THEOREM.

50. *If two triangles have three sides of the one equal respectively to three sides of the other, the two triangles are equal.*

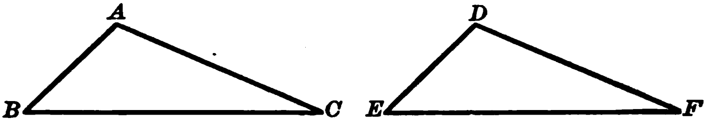


FIG. 29.

Let ABC and DEF be two triangles having the sides AB, BC, CA of the one equal respectively to the sides DE, EF, FD of the other.

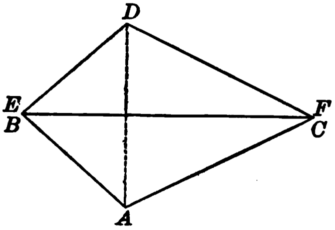


FIG. 30.

It is to be proved that the triangles are equal to each other.

Place the triangle ABC next to triangle DEF in such a way that BC shall coincide with its equal EF, and that BA shall fall at the same end of EF that ED does.

Draw DA.

CASE I. When DA intersects EF.

In the triangle EDA

side ED = side EA. [*By hypothesis.*

That is, the triangle is isosceles.

Therefore angle EDA = angle EAD; because the angles at the base of an isosceles triangle are equal to each other.

Also in the triangle FDA

side FD = side FA. [?

Therefore angle FDA = angle FAD. [?

Then

angle EDA + angle FDA = angle EAD + angle FAD. [?

That is, angle EDF = angle EAF, or
= angle BAC.

That is, the triangles ABC, DEF have two sides and the included angle of the one equal respectively to two sides and the included angle of the other.

Therefore they are equal; because if two triangles have two sides and the included, etc.

In a similar way triangles ABC and DEF may be proved equal

II. When DA intersects EF produced;

III. When DA intersects EF at E or at F.

Therefore, *If two triangles have three sides of the one equal, etc.* Q.E.D.

51. Defs. A **converse** of a given theorem is a second theorem formed from the first by interchanging the conclusion and a condition. Thus the converse of Prop. III is:

"*If a triangle has two angles equal to each other,* ***then the sides opposite those angles are equal to each other.***"

It must be noticed that when a given theorem has several conditions it will have several converses.

The given theorem is called the *direct* theorem. Every good definition is true both directly and conversely; that is, the definition stands for two propositions. Thus § 33 means:

If a triangle has two sides equal, ***it is isosceles***;
If a triangle is isosceles, ***it has two sides equal.***

Exercise. What is the converse of Prop. IV?

In the figure for Prop. IV let EF and DA intersect at O. Prove triangles EDO and EAO equal. (One proof uses Prop. II.)

Give the equal homologous parts resulting.

What is the relation of *EO* to *DA*?

Note. The two following theorems are useful in establishing several of the problems in §§ 53–58. § 52*a* is often assumed, and § 52*b* is included in a more general proposition proved later.

52*a*. Theorem. If a straight line through any point within a circle is indefinitely prolonged, it will intersect the circumference twice.

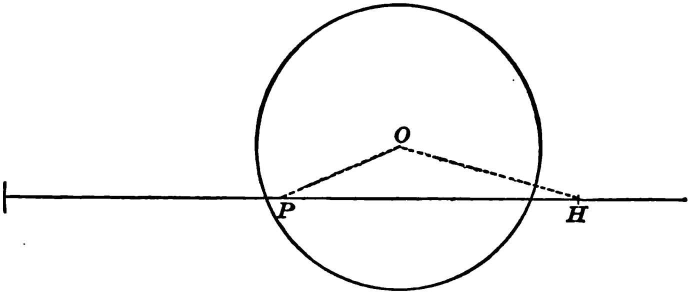


Fig. 31.

Lay off $PH =$ two radii.

Draw *PO*, *OH*.

$$\begin{aligned} PO + OH &> PH; \\ OH &> PH - OP \\ &> PH - \text{radius} \\ &> \text{radius}. \end{aligned}$$

$\therefore$ *PH* has a point (*P*) within the circumference and a point (*H*) outside the circumference.

$\therefore$ *PH* intersects the circumference.

Show in the same way that another part of the line intersects the circumference.

52*b*. Theorem. If the distance between the centers of two equal circles is less than two radii, the circles intersect.

This is proved by showing that circle I has a point within circle II and another point outside circle II.

PROPOSITION A. PROBLEM.

53. *To bisect a given straight line.*

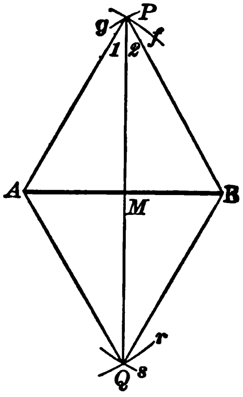


FIG. 32.

Let AB be a given straight line. It is required to bisect it.

Construction. With A and B as centers and with a radius greater than one half of AB, draw the arcs f, g intersecting at P.

And with A and B as centers and with a radius greater than one half AB, draw the arcs r, s intersecting at Q.

Draw PQ intersecting AB at M.

Then AB is bisected at M.

Proof. Draw AP, BP, AQ, BQ.

The triangles APQ, BPQ have three sides of one equal respectively to three sides of the other.

Therefore they are equal; because [§ 50.

Therefore angle P_1 = angle P_2. [§ 47.

Then the triangles PAM, PBM are equal, because [?

Therefore side AM = side BM. [?

That is, AB is bisected at M. Q.E.F.

53*a*. COR. *A given line has but one mid-point.*

54. SCH. Observe that the treatment of a problem consists of *Construction* and *Proof.*

The construction should proceed as directly as possible to the required result. The proof—like the proof of a theorem—may require further auxiliary construction. There may be in a problem three or four kinds of lines,—given lines, construction lines, required lines, lines to aid in proof.

EXERCISES. 1. What is the relation of PM to AB?

2. Show that if C is any point on PQ, $CA = CB$.

PROPOSITION B. PROBLEM.

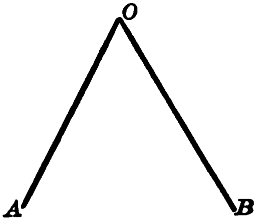


FIG. 33.

55. *To bisect a given angle.*

Let AOB be a given angle.

It is required to bisect it.

With O as center and any radius, describe arcs intersecting OA, OB at P, Q.

With P, Q as centers and radius greater than one half the straight line PQ, describe arcs intersecting at C.

Draw OC.

Then angle AOB is bisected by OC.

Draw PC, QC.

[Prove angle O_1 = angle O_2.]

55*a*. COR. *An angle has but one bisector.*

PROPOSITION C. PROBLEM.

56. *To draw a perpendicular to a line at a given point in the line.*

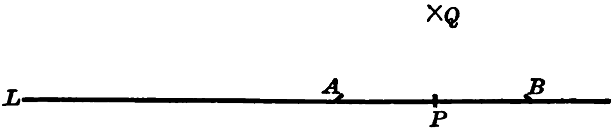


FIG. 43.

Let L be a given line, and let P be a given point in the line.

It is required to draw a perpendicular to L at P.

With P as a center and any radius, lay off $PA = PB$.

With A and B as centers and any radius greater than one half $\overline{AB}$, describe arcs f, g intersecting at Q.

Draw QP.

Then PQ is the perpendicular required.

Draw AQ, BQ. [Prove $\angle P_1 = \angle P_2$ by Prop. IV.]

Note. The *data* of a problem are the elements furnished (or assumed) for the construction of the figure. The data of a problem correspond to the hypothesis of a theorem.

The *quæsita* of a problem are the elements sought in the construction; they correspond to the conclusion of a theorem.

Give the data of Problems A, B, and C.

Make a theorem about an isosceles triangle which shall correspond to Problem C.

PROPOSITION D. PROBLEM.

57. *To draw a perpendicular to a line from a given point outside the line.*

Let L be a given line, and let A be a given point outside the line.

It is required to draw a perpendicular to L from A.

With A as center and with radius as long as from A to some point on the opposite side of L, describe an arc cutting L at P, Q. [*Post.*

With P and Q as centers and a radius longer than one half PQ, describe arcs f, g intersecting at S.

Draw AS, intersecting L at T.

Then AT is the required perpendicular to L from A.

Draw AP, AQ, PS, QS.

[Prove the triangles APS, AQS equal by § 50.]

Therefore angle $A_1 =$ angle A_2.

Then the triangles APT, AQT are equal. [§ 44.

Therefore angle $T_1 =$ angle T_2, etc.

Exercises. 1. To divide a given line into four equal parts.

2. To construct an isosceles triangle when the base and one side are given.

3. If a line is drawn from the vertex of an isosceles triangle bisecting the base, it is perpendicular to the base.

PROPOSITION E. PROBLEM.

58. *At a given point in a given straight line to construct an angle equal to a given angle.*

Let P be the given point on the given straight line L, and let A be the given angle.

It is required to construct at P on L an angle equal to A.

With A as a center and any radius, describe an arc intersecting the sides of A at B, C.

With P as a center and the same radius, describe an arc g intersecting the line L at Q.

With Q as a center and a radius equal to the straight line BC, describe an arc intersecting g at R.

Draw PR.

Then RPQ is the angle required.

Draw QR and BC.

Then the triangles ABC and PQR have——

Therefore they are equal; because [§ 50.

Therefore angle A = angle RPQ. [?

59. SCH. Observe that there are four positions that RPQ may have; and that the problem may be stated in different form—such as:

To construct an angle equal to a given angle.

EXERCISES. 1. To construct a triangle when two sides and the included angle are given.

2. To construct a triangle equal to a given triangle.

3. In the isosceles triangle ABC, $AB = AC$. Let H be the mid-point of AB, and K be the mid-point of AC. Prove $CH = BK$.

4. In the triangle of ex. 3 suppose that the bisector of $\angle B$ meets AC at X, and that the bisector of $\angle C$ meets AB at Y. Prove $BX = CY$.

PROPOSITION V. THEOREM.

60. *If one straight line meet another, the sum of the two adjacent angles is equal to two right angles.*

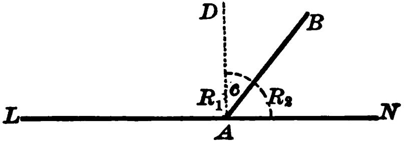


FIG. 35.

Let the line AB meet the line LN at A.

It is to be proved that angle $LAB +$ angle $BAN =$ two right angles.

At A draw AD perpendicular to LN. [?

Call angle LAD, R_1; angle DAN, R_2; angle DAB, c.

$$\text{Angle } LAB = R_1 + c,$$
$$\text{angle } BAN = R_2 - c.$$

$\therefore$ angle $LAB +$ angle $BAN = R_1 + R_2$. [*Gen. Ax.*

That is, angle $LAB +$ angle $BAN =$ two right angles.

Therefore, *If one straight line, etc.*

61. COR. 1. *At a point on a line only one perpendicular to the line can be drawn.*

Let LN be the line, and let p be a perpendicular at A.

It is to be proved that any other line at A (x) is not perpendicular to LN.

Call the angle between p and x angle c.

Then x makes *un*equal angles with LN. [?

$\therefore$ x is *not* perpendicular. [?

But x is *any* line other than p. [*Hyp.*

$\therefore$ *At a point on a line, etc.*

62. COR. 2. *All right angles are equal.*

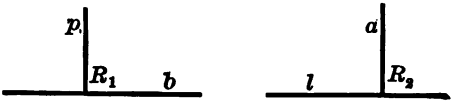


FIG. 36.

[Superpose and use § 61.]

63. Defs. Two angles are **supplements** of each other when their sum is two right angles.

Supplementary-adjacent angles are angles which are supplementary and adjacent.

63*a*. (Equivalent to § 60.) *If one straight line meet another, each of the angles formed is the supplement of the other.*

64. Cor. 3. *Supplements of the same angle or of equal angles are equal.*

65. Def. Two angles are **complements** of each other when their sum is one right angle.

66. Cor. 4. *Complements of the same angle or of equal angles are equal.*

Note. For practical purposes of measurement and comparison of angles the right angle is divided into ninety equal parts, called degrees; a degree is divided into sixty equal parts, called minutes; a minute is divided into sixty equal parts, called seconds.

Sixty seconds (60″) = one minute (1′).
Sixty minutes (60′) = one degree (1°).
Ninety degrees (90°) = one right angle (1|R).

Exs. 1. What is the supplement of 60°; of 100°? How many degrees in ⅓|R; in 1½|R; in 1⅔|R?

2. How many degrees in the supplement of ⅓|R? How many |R in 150°? How many |R in the supplement of 80°?

3. What is the complement of 20°? Supplement? What is the complement (and supplement) of ⅓|R?

4. How large is the angle whose supplement is equal to three times its complement?

5. What is the angle between the bisectors of two suppl-adj. angles?

6. What is the angle between the bisectors of two compl-adj. angles?

7. If one line is perpendicular to another, then the second line is perpendicular to the first.

PROPOSITION VI. THEOREM.

67. *If the sum of two adjacent angles is equal to two right angles, their exterior sides lie in the same straight line.*

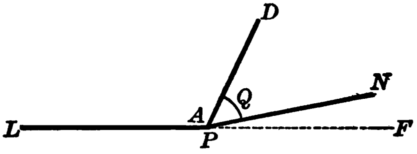


FIG. 37.

Let angle *LPD* and angle *DPN* be two adjacent angles, and let their sum equal two right angles.

It is to be proved that their exterior sides *LP, PN* are in a straight line.

If *PN* is not in the line *LP*, extend *LP* to *F*.

Call angle *LPD, A*; and angle *DPN, Q*.

Angle A + angle Q = two right angles. [?

Angle A + angle DPF = two right angles. [?

∴ angle A + angle Q = angle A + angle DPF. [?

∴ angle Q = angle DPF. [?

But this is impossible. [?

Therefore *PN* must be in the line *LP*.

68. SCH. This method of proof is called *reductio ad absurdum*. In this proposition it amounts, in brief, to this: If *LP* and *PN* are not in the same line, we can have two divergent lines from *P*, (1) *PN* (forming a suppl-adj. angle), and (2) *PF* (forming an extension of *LP*).

The existence of the *two* lines inevitably leads to absurdity. Therefore there can be only one; that is, the extension of *LP* and the line *PN* are the same thing.

This method is also called *the indirect method* of proof.

69. DEFS. A **straight angle** is the angle formed by two sides in the same straight line on opposite sides of the vertex. Angle LPF (Fig. 38) is a straight angle.

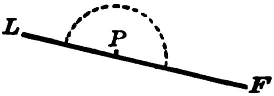


FIG. 38.

An **acute angle** (Fig. 39) is less than a right angle. An **obtuse angle** (Fig. 40) is greater than a right angle and less than two right angles. An **oblique angle** is either acute or obtuse.

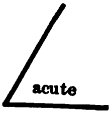


FIG. 39.

An **oblique line** is a line which meets another line making an oblique angle.

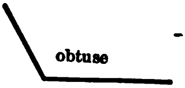


FIG. 40.

PROPOSITION VII. THEOREM.

70. *From a point outside a straight line only one perpendicular can be drawn to the line.*

Let AB be the straight line, P the point, PD the perpendicular from P to AB.

It is to be proved that any second line from P to AB—say PF—is *not* perpendicular to AB.

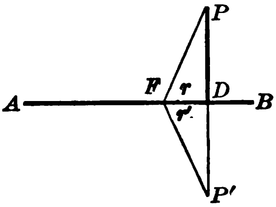


FIG. 41.

If possible let PF be a second perpendicular from P to AB.

Extend PD to P', making $DP' = PD$.

Draw $P'F$.

[Prove the triangle PDF = triangle $P'DF$.]

[Then angle r = angle r' = a right angle;
$r + r'$ = two right angles.]

Therefore PFP' is a straight line; because [§ 67.

But this is absurd; because (Ax. I).

Therefore PF is not perpendicular to AB.

That is, no other line than PD can be a perpendicular from P to AB.

Therefore, *From a point, etc.*

QUESTIONS. *What* is absurd?

What is true of *r* and *r'* whether *PFP'* is straight or not?

If *PFP'* is *not* straight, what is true of *r*?

Go through the argument in reverse order, beginning at the *absurd* point.

PROPOSITION VIII. THEOREM.

71. *If two straight lines intersect, the vertical angles are equal.*

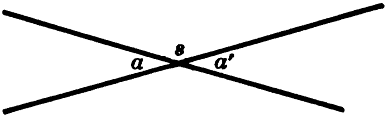


FIG. 42.

EXERCISES. 1. If *a* and *a'* are bisected, what is true of those bisectors?

2. If a straight line bisects *a*, what else is true of that bisector?

3. If *a* and *s* are bisected —— ?

4. Converse of ex. 3.

5. Prove Prop. VI, using § 64.

PROPOSITION IX. THEOREM.

72. *The perpendicular from a point to a straight line is the shortest line from the point to the line.*

[*PFP'* is *longer* than *PDP'*.

∴ *PF* is longer than *PD*.

State this theorem in typical form. What is its converse? Prove the converse.

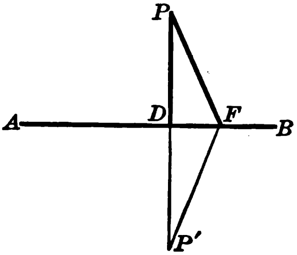


FIG. 43.

72a. DEF. The **distance** of a point from a line is the length of the perpendicular from the point to the line.

73. DEF. The **altitude** of a triangle is the perpendicular from the vertex to the base (or to the base produced).

PROPOSITION X. THEOREM.

74. *If two angles of a triangle are equal, the sides opposite are equal.*

Let ABC be a triangle in which angle B = angle C.

It is to be proved that side AC = side AB.

If possible let AB be greater than AC.

From B on BA lay off $BD = CA$.

Draw CD.

Then the triangles ACB and DCB have, etc.

Therefore they are equal; because [§ 44.

But this is absurd. [§ 42, (5).

Therefore side AB cannot be greater than side AC.

In the same way it can be shown that AB is not less than AC.

Therefore if AB is not greater than AC, and if it is not less than AC, then $AB = AC$. [?

Therefore, etc.

Note. This method of proof is called *exclusion.*

PROPOSITION XI. THEOREM.

75. *If two angles of a triangle are unequal, the side opposite the greater angle is greater than the side opposite the less angle.*

Let ABC be a triangle, and let angle ABC (call it B) be greater than angle A.

It is to be proved that side AC is greater than side BC.

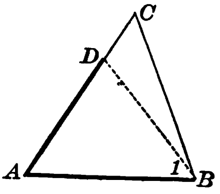


Fig. 44.

At B on the line AB, and on the same side of it as is the angle ABC, construct an angle ABD (call it B_1) equal to angle . [?

Then triangle ABD is isosceles. [?

Then BC is less than $BD + DC$ (?) and so is less than AC.

PROPOSITION XII. THEOREM.

76. *If two sides of a triangle are unequal, the angle opposite the greater side is greater than the angle opposite the less side.*

Let ABC be a triangle, and let AC be greater than AB.

It is to be proved, etc.

(1) If possible let angle B = angle C. [*Use* § 74.

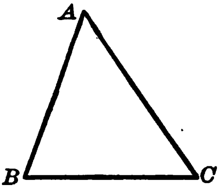


FIG. 45.

(2) If possible let angle C be greater than angle B. [*Use* § 75.

[Then B is greater than C.]

76*a*. THEOREM. *The altitude of an isosceles triangle bisects the base.*

[See ex. 3, page 23.]

Ex. *What other properties does the altitude of an isosceles triangle have?*

77. DEFS. An **equilateral** triangle is one in which all the sides are equal.

An **equiangular** triangle is one in which all the angles are equal.

EXS. 1. *An equilateral triangle is equiangular.*

2. *Converse of ex. 1.*

77*a*. DEF. The **median** of a triangle is the line from the vertex to the mid-point of the base.

EXS. 1. Restate ex. 3, page 23, using §§ 73 and 77*a*.

2. Restate § 76*a*.

PROPOSITION XIII. THEOREM.

78. *If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first is greater than the third side of the second.*

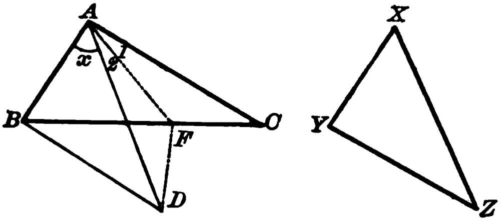


FIG. 46.

Let the triangles ABC and XYZ have $AB = XY$, $AC = XZ$, angle BAC greater than angle X.

It is to be proved that BC is greater than YZ.

At A on BA, and on same side as is angle BAC, construct an angle equal to X, [?

and make $AD = XZ$. [?

Draw BD. [?

Then $BD = YZ$ (?), and we are to prove BC greater than BD.

Bisect angle CAD by AF. [?

Draw FD. [?

[Prove triangles DAF, CAF equal.]

Then $FD = FC$. [?

BD is less than BFD, and so
less than BFC.

That is, YZ is less than BC.

PROPOSITION XIV. THEOREM.

79. *If two triangles have two sides of one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, then the included angle of the first is greater than the included angle of the second.*

Let ABC, XYZ be the triangles

$AB = XY$, $AC = XZ$, BC greater than YZ.

Then A is greater than X. [Use *exclusion.*

80. THEOREM. *The line bisecting the vertical angle of an isosceles triangle is perpendicular to the base.*

80*a*. THEOREM. *The line bisecting the vertical angle of an isosceles triangle bisects the base.*

80*b*. THEOREM. *The perpendicular to the base of an isosceles triangle at the middle point of the base passes through the vertex.*

[Show that this perpendicular bisector is identical with the angle bisector of §§ 80, 80*a*.]

81. DEF. The **hypotenuse** of a right triangle is the side opposite the right angle.

"Side" in a right triangle often means "side not hypotenuse."

81*a*. THEOREM. *Two right triangles are equal if the hypotenuse and an adjacent angle of one are equal respectively to the hypotenuse and an adjacent angle of the other.*

[Superpose, and use § 70.]

82. THEOREM. *If two oblique lines from a point to a line meet the line at equal distances from the foot of the perpendicular, the two oblique lines are equal.*

PROPOSITION XV. THEOREM.

83. *Two right triangles are equal when the hypotenuse and a side of one are equal respectively to the hypotenuse and side of the other.*

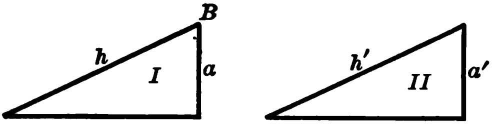


FIG. 47.

Let $a = a'$, $h = h'$, etc.

Place the triangles adjacent, with B' at B, and H' at H. Then ABA' forms an isosceles triangle. [?

$\therefore\ b = b'$. [§ 76*a*.

$\therefore\ I = II$.

83*a*. COR. *Converse of* § 82.

PROPOSITION XVI. THEOREM.

84. *If two oblique lines from a point to a line meet that line at unequal distances from the foot of the perpendicular, the more remote is the greater.*

It is to be proved that PA is greater than PB.

Extend PD to Q, making $DQ = PD$.

Draw QA, QB, extending QB to F.

PAQ is greater than PFQ, [?
which " " " PBQ. [?

PAQ is $= 2PA$; $PBQ = ?$

$\therefore$ PA is greater than PB.

85. COR. *Converse of* § 84.

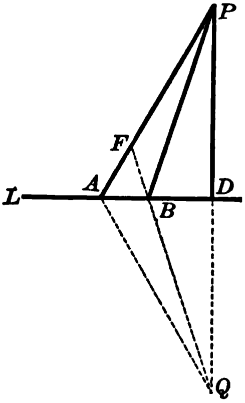


FIG. 48.

[*Indirect method.*

86. Sch. Triangles are classified—

I. According to the equality of their sides:

(1) Scalene—No two sides equal.
(2) Isosceles—Two sides equal.
(3) Equilateral—Three sides equal.

II. According to the size of one angle as compared with a right angle:

(1) Acute-angled—Every angle acute.
(2) Right-angled—One right angle.
(3) Obtuse-angled—One obtuse angle.

87. Sch. A figure is *determined* when the parts given are such that one figure can be formed with those parts and no different figure can be formed. Plainly, when we have proved that two figures are equal throughout if certain parts are equal, we have proved that those parts determine the figure.

An *ambiguous* case of a construction occurs when two constructions are possible with the *data.*

88. Probs. Construct the triangle when there are given—

(*a*) Two sides and the included angle.
(*b*) Two angles and the included side.
(*c*) Two sides and an angle opposite one of the sides.
(*d*) Three sides.

Construct the *right* triangle when there are given—

(*e*) Hypotenuse and one adjacent angle.
(*f*) Two *sides.*
(*g*) One side and adjacent acute angle.
(*h*) Hypotenuse and one side.

89. Theorem. *Any side of a triangle is greater than the difference of the other two sides.*

89*a*. (Equivalent to Ax. II.) *Any side of a triangle is less than the sum of the other two sides.*

Exercises. 1. In § 88, (1) Is problem (*a*) always possible?
(2) Is problem (*c*) always possible?
(3) Is problem (*c*) always determinate?
(4) Is problem (*d*) always possible?
(5) Problem (*g*) is included in what earlier problem?

2. *The sum of all the angles about a point is equal to four right angles.*

3. *Only two equal oblique lines can be drawn from a point to a straight line.*

4. The altitudes to the sides of an isosceles triangle are equal.

5. If from any point within a triangle lines are drawn to the vertices, their sum is greater than one half the sum of the sides of the triangle.

6. If from any point within a triangle lines are drawn to the extremities of the base, their sum is less than the sum of the two sides of the triangle.

7. If from any point within a triangle lines are drawn to the vertices, their sum is less than the sum of the three sides of the triangle.

8. If a point is in the perpendicular bisector of a line, it is equally distant from the extremities of the line. (See ex. 2, page 21.)

9. If a point is outside the perpendicular bisector of a line, it is unequally distant from the extremities of the line.

10. If a point is in the bisector of an angle, it is equally distant from the sides of the angle.

11. State and prove the converse of ex. 8.

12. State and prove the converse of ex. 4.

13. The median of a scalene triangle is not the altitude.

14. Prove § 83 by placing *I* adjacent to *II*, with *AB* on *A'B'*. Draw *HH'*.

PARALLELS AND PARALLELOGRAMS.

PROPOSITION XVII. THEOREM.

90. *Two lines perpendicular to the same straight line are parallel.*

Let AB, CD be perpendicular to LN at A, C.

It is to be proved that AB and CD are parallel.

If they are not parallel, they will meet at some point—say at X. [*Use* § 70.

PROPOSITION F. PROBLEM.

91. *To draw through a given point a parallel to a given line.*

PROPOSITION XVIII. THEOREM.

92. *If a line is perpendicular to one of two parallels, it is perpendicular to the other parallel also.*

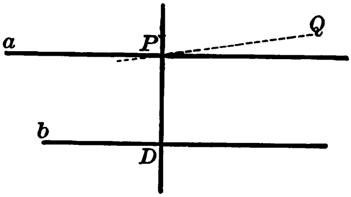


FIG. 49.

Let a and b be two parallel lines, and let PD be perpendicular to b.

It is to be proved that PD is perpendicular to a.

Draw PQ perpendicular to PD. [*Use the indirect method*

92*a*. PROBLEM. *To draw from any point a perpendicular to two parallels.*

93. THEOREM. *Two lines parallel to the same straight line are parallel to each other.* [*Use method of* § 90.

94. DEFS. **A transversal** is a line which intersects two other lines.

Eight angles are formed by a transversal and are named as follows:

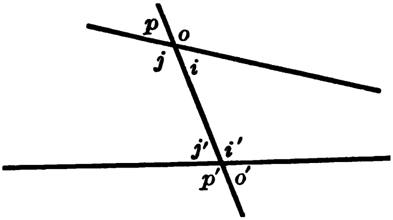


FIG. 50.

(1) **Alternate-interior** angles are on opposite sides of the transversal, within the two lines. *i* and *j′* are alternate interior; *j* and *i′* are alt.-int.

(2) **Alternate-exterior** angles are on opposite sides of the transversal, outside the two lines. *o* and *p′* are alt.-ext.; *p* and *o′* are alt.-ext.

(3) **Corresponding** angles are on the same side of the transversal, each on the same side of its line. *o* and *i′*; *j* and *p′*.

PROPOSITION XIX. THEOREM.

95. *If two straight lines are cut by a third, making the alternate-interior angles equal, the two lines are parallel.*

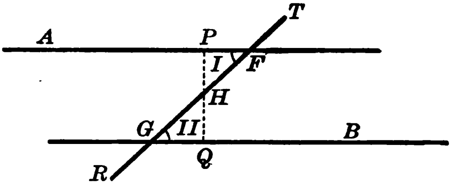


FIG. 51.

Let *A* and *B* be two lines cut by *TR* at *F* and *G*, making the alt.-int. angles *AFG, FGB* equal.

It is to be proved that *A* and *B* are parallel.

Bisect *FG* (at *H*). Through *H* draw *PQ* perpendicular to *B*.

Then the two triangles *I* and *II* are equal. [?

Then *A* is also perpendicular to *PQ*. [?

That is, *A* and *B* are perpendicular to the *same* straight line *PQ*, etc.

Suppose the angles at *F* and *G* had been right?

Suppose the other pair of alt.-int. angles had been taken?

NOTE. The theorem § 95 may be stated: *If the alternate-interior angles on a transversal are equal, the two lines are parallel.*

95*a*. COR. 1. *If two straight lines are cut by a transversal making the alternate-exterior angles equal, the lines are parallel.*

96. COR. 2. *If two straight lines are cut by a transversal making the corresponding angles equal, the lines are parallel.*

97. COR. 3. *If the sum of the two interior angles on the same side of the transversal is equal to two right angles, the two lines are parallel.*

PROPOSITION XX. THEOREM.

98. *If two parallel lines are cut by a transversal, the alternate-interior angles are equal.*

[Prove from preceding figure. Prove triangles *I* and *II* equal by § 81*a*.]

[Also prove by indirect method.]

98*a*. COR. 1. *Converse of* § 95*a*.

99. COR. 2. *Converse of* § 96.

100. COR. 3. *Converse of* § 97.

PROPOSITION XXI. THEOREM.

101. *The sum of the angles of a triangle is equal to two right angles.*

[Draw through *A* a parallel to *BC* and use § 98.]

102. COR. *If a side of a triangle be produced, the exterior angle equals what?*

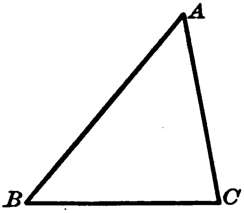


FIG. 52.

Why cannot a triangle have two right angles?

What is the relation of the two acute angles in a right triangle?

EXERCISE. *Two right triangles are equal when a side and acute angle of one are equal respectively to a side and homologous acute angle of the other.*

103. DEFS. A **quadrilateral** is a figure bounded by four straight lines. The **sides** of the figure are the lines which bound it. The **angles** are the angles between its sides. The **vertices** are the vertices of its angles.

A **diagonal** is a straight line joining any two vertices not consecutive.

It is assumed in this definition that every interior angle is less than a straight angle. Such a figure is a convex figure.

104. DEFS. (1) A **parallelogram** is a quadrilateral whose opposite sides are parallel. The bases are either pair of parallel sides.

(2) A **trapezoid** is a quadrilateral two of whose sides are parallel. The **bases** are the parallel sides.

(3) A **trapezium** is a quadrilateral no two of whose sides are parallel.

PROPOSITION XXII. THEOREM.

105. *The opposite sides of a parallelogram are equal.*

[Draw a diagonal and prove the triangles equal.]

106. THEOREM. *The opposite angles of a parallelogram are equal.*

If a parallelogram has one right angle, what is inferred from § 100?

If a parallelogram has two consecutive sides equal, what is inferred from § 105?

107. DEFS. (1) A **rectangle** is a parallelogram in which all the angles are right angles.

(2) A **rhombus** is a parallelogram in which all the sides are equal.

(3) A **square** is a rectangle in which all the sides are equal.

PROPOSITION XXIII. THEOREM.

108. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*

PROPOSITION XXIV. THEOREM.

109. *If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*

PROPOSITION XXV. THEOREM.

110. *The diagonals of a parallelogram bisect each other.*

Ex. *If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*

111. DEF. The **altitude** of a parallelogram (or of a trapezoid) is the perpendicular to a base from a vertex opposite that base.

EXS. 1. *Two parallels are everywhere equally distant.*

2. *The diagonals of a rectangle are equal.*

3. *The diagonals of a rhombus are perpendicular.*

4. *If each of two intersecting lines is parallel to each of two other intersecting lines, the angles between the first pair are equal to the homologous angles between the second pair.*

NOTE. In exs. 4 and 5 suppose a, b are intersecting lines and are parallel (or perpendicular) respectively to a' and b'. The point of intersection of a and b is sometimes called the point ab; and the angle between the lines is called the angle ab, meaning the angle which a must turn through to reach b. If the angle ab is taken "clockwise" (§§ 20, 21), then its homolog $a'b'$ must be taken clockwise.

5. *If each of two intersecting lines is perpendicular to each of two other intersecting lines, the angles between the first pair are equal to the homologous angles between the second pair.*

[Through the point ab draw $x \perp$ to a, $y \perp$ to b, and prove $\angle\, ab = \angle\, xy = \angle\, a'b'$.]

112. PROBLEM. *To construct a parallelogram when two sides and the included angle are given.*

113. PROBLEM. *To construct a rectangle with a given base and altitude.*

114. PROBLEM. *To construct a square on a given side.*

PROPOSITION XXVI. THEOREM.

115. *If two parallelograms have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the parallelograms are equal.*

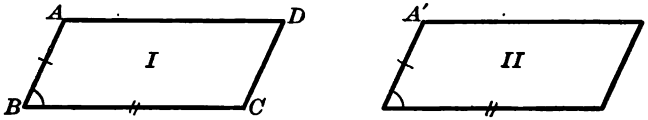


FIG. 53.

[Superpose, and use Ax. III.]

116. COR. *If two rectangles have the base and altitude of one equal respectively to the base and altitude of the other, the rectangles are equal.*

EXERCISES. 1. If the diagonals of a parallelogram are equal, the figure is a rectangle.

2. In an oblique parallelogram the diagonals are unequal. Which is the larger?

3. The sum of all the angles of a parallelogram is equal to four right angles.

4. If the opposite angles of a quadrilateral are equal, the figure is a parallelogram. (Prove two consecutive angles supplementary.)

MISCELLANEOUS PROPOSITIONS.

117. Defs. A **polygon** is a figure bounded by straight lines.

(1) A polygon of three sides and (2) a polygon of four sides have already been named.

(3) A polygon of five sides is a pentagon.
(4) " " " six " " " hexagon.
(5) " " " eight " " an octagon.
(6) " " " ten " " a decagon.
(7) " " " twelve " " " dodecagon.
(8) " " " fifteen " " " pentedecagon (or quindecagon).

A few other polygons have names derived from the Greek numerals, but such names are not in common use.

A polygon is assumed to be convex.

The **perimeter** of a polygon is the sum of its sides.

Exercises. 1. The sum of all the angles of a pentagon is equal to six right angles. (Draw diagonals from one vertex.)

2. The bisectors of two opposite angles of a parallelogram are parallel.

Note. Σ is used to denote "The sum of all terms like." Thus $\Sigma(a)$ means $a + b + c + \ldots$

3. In a hexagon $\Sigma(A) = 8\underline{|R}$

PROPOSITION XXVII. THEOREM.

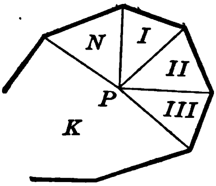


FIG. 54.

118. *The sum of the angles of a polygon is equal to two right angles multiplied by the number of sides of the polygon, less four right angles.*

Let K be a polygon of n sides.

It is to be proved that the sum of the angles in K

$$= \overline{(2 \text{ right angles}) \times n} - 4 \text{ right angles.}$$

Let P be any point within K.

From P draw lines to the vertices forming triangles *I, II,* etc. . . . *N.*

The sum of the angles in any one triangle equals two right angles. [?

∴ The sum of the angles in all the triangles equals $\overline{\text{two right angles}} \times n$. [?

The sum of the angles about P equals four right angles.

∴ The sum of the interior angles of the polygon equals $\overline{\text{two right angles} \times n}$ − four right angles. [?

$$\Sigma(A) = n\,.\,2|\underline{R} - 4|\underline{R}.$$

EXERCISE. *The sum of the exterior angles of a polygon formed by producing one side at every vertex is equal to four right angles.*

119. DEF. The **right bisector** of a line is the perpendicular erected at its middle point.

PROPOSITION XXVIII. THEOREM.

120. I. *Any point in the right bisector of a line is equally distant from the extremities of the line.*

II. *Any point outside the right bisector of a line is unequally distant from the extremities of the line.*

PROPOSITION XXIX. THEOREM.

121. I. *Any point in the bisector of an angle is equally distant from the sides of the angle.*

II. *Any point outside the bisector of an angle is unequally distant from the sides of the angle.*

122. SCH. The theorems § 121, I and § 121, II are each the *obverse* of the other. Evidently the same number of *obverses* may be made from a given direct theorem as there are conditions in its hypothesis. In this section attention will be given to the changes in only a single condition. Consider § 121, I as *the direct theorem;* we may then arrange the following set of theorems in which (II) is *the obverse* of (I), (III) is *the converse* of (I), and (IV) is the *contra-positive* of (I). It should be noticed that *the contra-positive* is the same theorem as *the direct* theorem in another form.

(I) THE THEOREM. Any point in the bisector of an angle — is equally distant from the sides.

(II) THE OBVERSE. Any point outside the bisector of an angle — is unequally distant from the sides.

(III) THE CONVERSE. Any point equally distant from the sides of an angle — lies in the bisector.

(IV) THE CONTRA-POSITIVE. Any point unequally distant from the sides of an angle — lies outside the bisector.

If we denote the hypothesis of § 121, I by X, and its conclusion by Y, these theorems may be stated:

(I) Every X is also Y.

(II) Every non-X is also non-Y.

(III) Every Y is also X.

(IV) Every non-Y is also non-X.

Observe in this scheme of theorems that when any pair not diagonally opposite are proved true, the truth of the whole set follows. It is sometimes convenient to use the following Rule in establishing a converse theorem:

123. The Rule of Identity. If there is but one X and but one Y, and if it is proved that X is Y, then it follows that Y is X.

Thus: In an isosceles triangle

There is but one median to the base, (one X)
There is but one right bisector of the base, (one Y)
The median is a right bisector, (X is Y)
Therefore the right bisector is the median. (Y is X)

124. THEO. *The line joining the middle points of two sides of a triangle is parallel to the third side.*

125. THEO. *Converse of* § 124.

126. THEO. *The line of* § 124 *is equal to one half the third side.*

127. THEO. *The line joining the middle points of the two non-parallel sides of a trapezoid is parallel to the bases.* [FG is a parallelogram.]

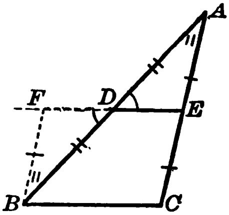


FIG. 55.

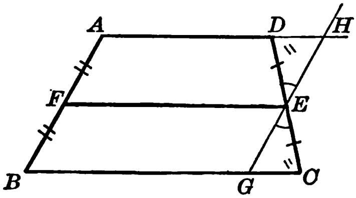


FIG. 56.

128. THEO. *Converse of* § 127.

128*a*. DEF. The **mid-parallel** of a trapezoid is the line joining the mid-points of the non-parallel sides.

128*b*. THEO. *The mid-parallel of a trapezoid is equal to one half the sum of the bases.* [Draw a diagonal.]

PROPOSITION G. PROBLEM.

129. *To divide a straight line into any number of equal parts.*

Let A ———————————————— B be the given line.

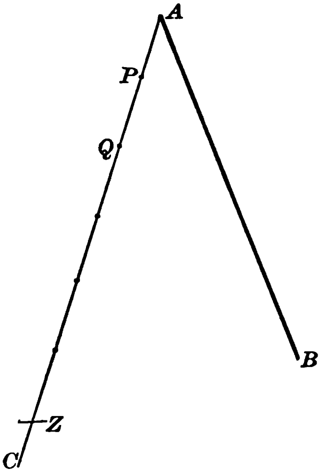


FIG. 57.

It is required to divide AB into n equal parts.
At A construct any angle BAC.
On AC lay off n equal parts AP, PQ, QR, etc.
Through the last point of division, Z, draw ZB.
Draw PP', QQ', etc., parallel to ZB.
Then P', Q', etc., divide AB into n equal parts.

For in the triangle AQQ', $AP' = P'Q'$. [?
In the trapezoid PR', $P'Q' = Q'R'$. [?

Exercise. The bisectors of two angles of a triangle intersect in a point which is on the bisector of the third angle.

130. Def. Three or more lines are **concurrent** when they intersect in a common point.

131. Theo. *The bisectors of the three angles of a triangle are concurrent.*

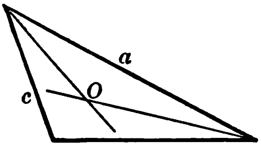


Fig. 58.

132. Theo. *The right bisectors of the three sides of a triangle are concurrent.*

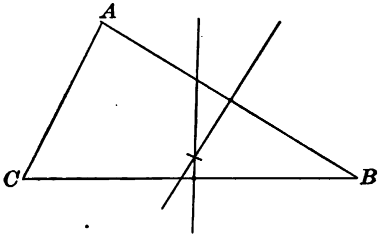


Fig. 59.

133. Theo. *The altitudes of a triangle are concurrent.*

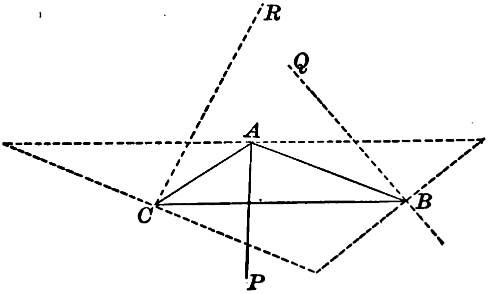


Fig. 60.

Exercise. In triangle ABC draw the medians AX, BY, intersecting at G. Bisect AG at H, and BG at K. Prove $HKXY$ a parallelogram. How does G divide AX? (See §§ 124, 126.)

134. Theo. *The medians of a triangle are concurrent.*

EXERCISES. 1. The bisector of two exterior angles (at A, B) of a triangle, and the bisector of the third interior angle (C), are concurrent.

2. If the triangle of § 132 is a right triangle, what occurs? (See § 125.)

If the triangle of § 133 is a right triangle, what occurs?

3. What is true of the points of concurrence of §§ 131, 132, 133 when the triangle is isosceles?

What is true when the triangle is equilateral?

4. *In the square $ABCD$, $BP = BC$, and PQ is perpendicular to BD. Prove $DP = PQ = QC$.*

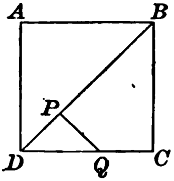


FIG. 61.

135. DEF. The **locus** of a point is the line, or the group of lines, which contains all the points, and only those points, that satisfy a given condition.

The locus of the points equally distant from the extremities of a line is the right bisector of the line; for the right bisector contains all the points equally distant from the extremities of the line (If a point is equally distant, etc.); and the right bisector contains no other points (If a point is in the right bisector, etc.).

136. SCH. In order to prove completely that a certain figure is the locus required, a direct and converse theorem—or else a direct and obverse theorem—involving the given condition and the required locus must be established.

EXERCISES. 1. The bisector of an angle is the locus of points equally distant from the sides of the angle.

2. What is the locus of points—

(1) At a given distance from a given point?
(2) At a given distance from a given line?
(3) Equally distant from two given points?
(4) Equally distant from two given lines—
  (*a*) When the lines intersect?
  (*b*) When the lines are parallel?

NOTE.—The phrases "The locus of a point" and "The locus of the points" are both used; but the first expression generally suggests that the locus is traced by a point which *moves* subject to some conditions.

EXERCISES.

1. The perpendicular from the extremity of the base of an isosceles triangle to the opposite side makes with the base an angle equal to one half the vertical angle.

2. The line joining the feet of the perpendiculars from the extremities of the base of an isosceles triangle to the opposite sides is parallel to the base.

3. Let CD be the altitude of the triangle ABC. Prove the difference of the angle C_1 and C_2 equal to the difference of the angles A and B.

4. The bisector of the exterior angle at the vertex of an isosceles triangle is parallel to the base.

5. In a right triangle the mid-point of the hypotenuse is equally distant from the vertices.

6. Converses of Ex. 5.

7. If the median of a triangle is greater than one half the base, the vertical angle is acute.

8. State and prove the theorem when the median is less than one half the base.

9. State and prove the converse of Ex. 7 and of Ex. 8.

10. Let ABC be an isosceles triangle with vertical angle C. Produce AC to D making $CD = CA$. Draw BD. Prove DBA a right triangle.

11. The bisectors of the four angles of a quadrilateral form a quadrilateral whose opposite angles are supplementary.

12. Prove the converse of § 106.

13. To construct a square when the diagonal is given.

14. To construct a rectangle when the diagonal and one side are given.

15. To construct a square when the sum of diagonal and one side is given. [See figure to Ex. 4 on page 50.

16. To construct a square when the difference of diagonal and one side is given.

17. How many elements must be given to construct a parallelogram? Make three problems for the construction of a parallelogram.

18. The perpendiculars from the extremities of the base to the median are equal.

19. The sum of the medians of a triangle is less than the perimeter of the triangle.

20. The median of a triangle is equal to, greater than, or less than one half the base according to the size of the vertical angle. Complete the theorem and prove it.

21. Two triangles (ABC, $A'BC$) have the same base, are on opposite sides of the base, and have equal altitudes (AD, $A'D'$). Prove that AA' is bisected by the base.

22. To construct an isosceles triangle given the altitude and the vertical angle.

23. To construct an isosceles triangle given the altitude and the perimeter.

24. To construct a triangle given one side, an adjacent angle, and the perimeter.

25. To construct a triangle given one side, an adjacent angle, and the difference of the other two sides.

26. Through a given point within an angle draw a line terminated by the sides of the angle and bisected by the point. State this as a problem involving a triangle and median.

27. What is the locus of the mid-point of a line from vertex to base of a triangle?

28. The sum of the medians of a triangle is greater than three fourths the perimeter.

29. The sum of the perpendiculars (PH, PK) from a movable point (P) in the base (AB) of an isosceles triangle (ABC) is equal to the perpendicular from one of the equal angles (A) to the opposite side (BC).

30. From any point (P) within an equilateral triangle (XYZ) the sum of the perpendiculars to the three sides is equal to an altitude of the triangle.

31. In the last two exercises let P move outside the triangle and generalize the theorems.

32. Let the mid-parallel (EF) of a trapezoid be intersected by the diagonals at X, Y. Discover a theorem about XY.

33. The bisector of the vertical angle of a triangle and the altitude form an angle equal to one half the difference of the base angles.

34. Construct the right triangle having $a = \frac{1}{2}h$. What is the relation of the acute angles?

35. State and prove the converses of Ex. 4.

36. To construct an isosceles triangle when the perimeter and one angle are given.

THE CIRCLE.

137. Def. A **central angle** is one whose vertex is at the center and whose sides are radii.

A central angle **intercepts** the arc between the radii.

138. Def. A **sector** is the part of a circle bounded by two radii and the intercepted arc.

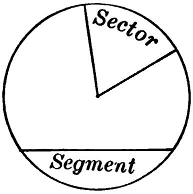


Fig. 62.

139. Defs. A **chord** is a straight line joining any two points of the circumference.

The chord **subtends** the arc whose extremities it joins.

140. Def. A **segment** is the part of a circle bounded by a chord and the subtended arc.

140*a*. Two radii form two central angles and intercept two conjugate arcs whenever the circumference is complete.

These intercepted arcs are distinguished as major and minor (§ 21); and similar language is used when referring to the sectors and segments associated with these arcs.

141. Def. A **diameter** is a chord through the center of the circle.

142. Cor. 1. *All diameters in a circle are equal.*

143. Cor. 2. *A diameter bisects the circle and also the circumference.* [Superpose.]

144. Def. A **semicircle** is one half a circle.

145. Def. A **semicircumference** is one half a circumference.

PROPOSITION XXX. THEOREM.

146. *The radius perpendicular to a chord bisects the chord.*

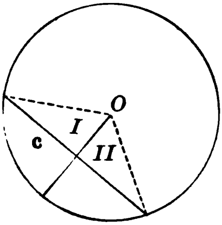


FIG. 63.

147. COR. *The right bisector of a chord passes through the center of the circle.*

EX. Given a circle and the middle point of a chord, to construct the chord.

148. THEOREM. *If two circles have equal radii, the circles are equal.*

Let *I* and *II* be two circles having radius of *I* equal to radius of *II*. It is to be proved that the circles are equal. Superpose *I* upon *II* with center O_1 on center O_2. Then will circumference C_1 coincide with C_2. [?

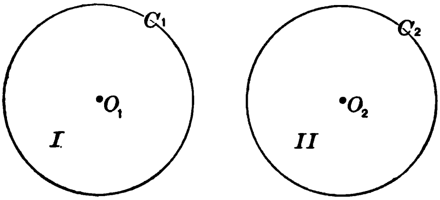


FIG. 64.

149. SCH. Strictly speaking, arcs are *equal* (§ 27) only when they are parts of equal circles (or of the same circle).

The definitions and explanation of addition and subtraction of magnitudes given in §§ 24, 25 are easily extended to arcs in equal circles (or in the same circle).

150. THEOREM. *In equal circles (or in the same circle) if two central angles are equal, the intercepted arcs are equal.*

150*a*. PROB. To construct an arc equal to a given arc.

151. COR. *The radius perpendicular to a chord bisects the subtended arc.*

Ex. To bisect a given arc.

152. THEOREM. *In equal circles, if two arcs are equal, the central angles intercepting them are equal.* [Superpose.]

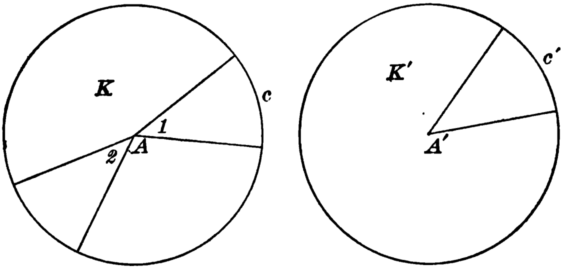


FIG. 65.

153. DEF. A **quadrant** is one fourth of a circumference.

154. THEOREM. *In equal circles, if two arcs are equal, the chords subtending them are equal.* [Prove triangles equal.]

155. THEOREM. *In equal circles, if two chords are equal, the subtended minor arcs are equal.* [Use § 150.]

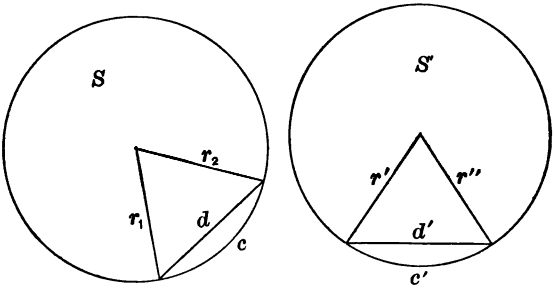


FIG. 66.

155*a*. PROBLEM. *To construct an arc equal to a given arc.* [Use § 155.]

156. THEOREM. *In equal circles, if two chords are equal, they are equally distant from the center.*

[Use triangle *OAD* in Fig. 67.]

157. THEOREM. *In equal circles, if two chords are equally distant from the center, the chords are equal.*

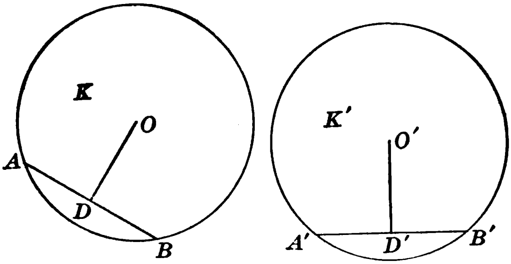


FIG. 67.

[§§ 156, 157 may also be proved by superposition.]

Ex. AB is a chord of given length moving within the given circle K. Find the locus of the middle point of AB.

158. THEOREM. *In equal circles, if two central angles are unequal, the greater angle intercepts the greater arc.*

159. THEOREM. *In equal circles, if two arcs are unequal, the greater arc is intercepted by the greater central angle.*

160. THEOREM. *In equal circles, if two minor arcs are unequal, the greater arc is subtended by the greater chord.*

[Use §§ 159, 78.]

161. THEOREM. *In equal circles, if two chords are unequal, the greater chord subtends the greater minor arc.*

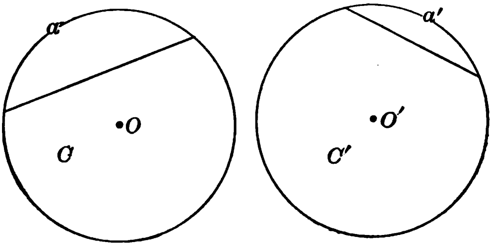


FIG. 68.

PROPOSITION XXXI. THEOREM.

162. *In equal circles, if two chords are unequal, the greater chord is nearer the center.*

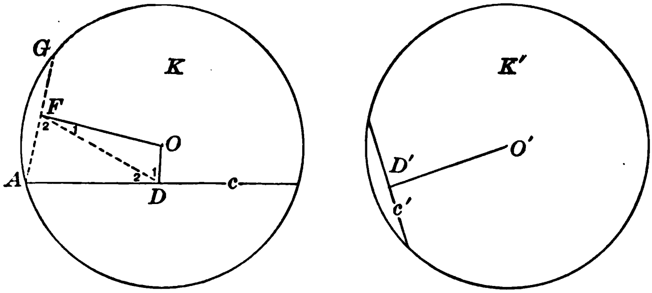


FIG. 69.

Let K and K' be equal circles, and let chord c be greater than chord c'.

It is to be proved that c is at less distance from the center than c' is.

Draw OD and $O'D'$, perpendiculars from center to chords.

In the major segment on c construct chord $AG = c'$.

Draw OF from O perpendicular to GA.

Draw FD.

Then $$FA = \frac{1}{2}\,GA\ [?] = \frac{1}{2}\,c';$$

and $$AD = \frac{1}{2}\,c.$$

In triangle FAD, FA is less than AD. [Gen. Ax.

$\therefore\ \angle D_2$ is less than $\angle F_2$. [Why?

But $\angle OFA = \angle ODA$; [?

$\therefore\ \angle D_1$ is greater than $\angle F_1$; [§ 42 (4).

$\therefore$ in triangle FOD side OF is greater than OD. [?

But OF equals $O'D'$.

$\therefore\ O'D'$ is greater than OD.

$\therefore\ c$ is at less distance from O than c' is from O'.

Therefore, *In equal circles, etc.* Q. E. D.

PROPOSITION XXXII. THEOREM.

163. *In equal circles, if two chords are unequally distant from the center, the one at less distance from the center is the greater chord.*

EXERCISES. 1. Through a given point within a circle construct the shortest chord.

2. If two chords of a circle intersect and make equal angles with the radius through the point of intersection, the two chords are equal.

3. If two equal chords of a circle intersect, each part of one chord is equal to a part of the other.

4. In a given circle construct a chord which shall equal a given line and be parallel to another given line.

5. In a given circle construct a chord which shall equal a given line and be perpendicular to another given line.

164. DEF. A **secant** is a straight line which intersects the circumference in two points.

165. THEOREM. *A straight line cannot intersect a circumference in more than two points.*

For if three intersections were possible, there would be three equal lines from center to these points. Therefore there would be two equal oblique lines cutting off unequal lengths from the foot of the perpendicular.

166. DEFS. A **tangent** to a circle is a straight line which touches the circumference in one point, and in only one even when it is prolonged.

The point of **contact**, or point of **tangency**, is the point common to the circumference and line.

PROPOSITION XXXIII. THEOREM.

167. *If a line is perpendicular to a radius at its extremity, the line is tangent to the circle.*

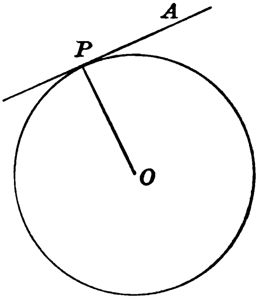


FIG. 70.

Let PA be perpendicular to the radius OP at the extremity of the radius.

Let A be any point in PA except P.

Draw OA.

OP is less than OA. [?

That is, A lies outside the circumference.

But A is *any* point in PA except P.

∴ every point in PA except P is outside the circumference.

∴ PA is tangent to the circle at P. [?

167*a*. PROBLEM. *To draw a tangent to a circle through a given point in the circumference.*

168. COR. 1. *The radius at the point of contact of a tangent is perpendicular to the tangent.*

[Suppose the radius an oblique line. Then a second oblique line equal to radius would fix a second point common to tangent and circle.]

169. COR. 2. *The perpendicular to a tangent at the point of contact passes through the center of the circle.*

EX. 1. If a radius and a tangent are perpendicular, they intersect at the point of contact.

2. TA is a tangent at T to a given circle K. TA is constant (that is, always the same length), but T moves about the circumference. Find the locus of A.

PROPOSITION XXXIV. THEOREM.

170. *Two parallel lines intercept equal arcs on the circumference.*

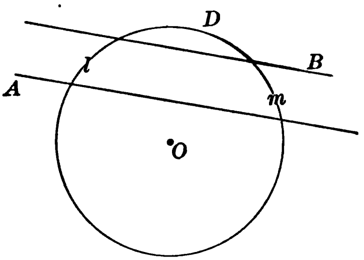


FIG. 71.

1st. Let A and B be parallel *secants* to the circle K; and let a and b be the arcs subtended by the chords of A and B respectively.

It is to be proved that the intercepted arcs l and m are equal.

Draw radius OD perpendicular to A and B.

$$\frac{a}{2} - \frac{b}{2} = l; \quad \frac{a}{2} - \frac{b}{2} = m.$$

2d. Let A be a secant, and let C be a parallel tangent.

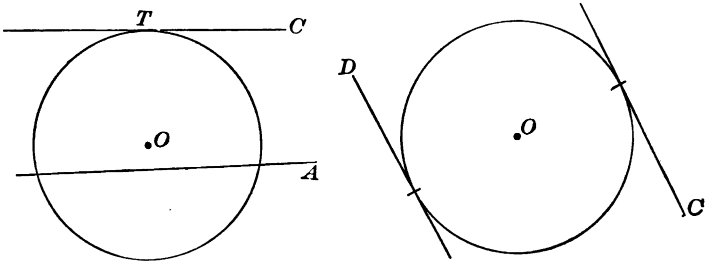


FIG. 72.

It is to be proved . . .

Draw OT.

3d. Let C and D be parallel tangents.

Draw secant A parallel to C.

Note. It should be observed that when an angle has its vertex within a circle, its sides *intercept* one arc; but that when an angle has its vertex outside the circle, and its sides either intersect or are tangent to the circumference, the angle intercepts two arcs.

171. Def. An **inscribed angle** is one whose vertex is on the circumference, and whose sides are chords.

Proposition XXXV. Theorem.

172. *An inscribed angle is equal to one-half the central angle which intercepts the same arc.*

1st. Let one side of the angle be a diameter.

In the circle K let A be an inscribed angle intercepting arc BC, and let AB be a diameter, O the center.

It is to be proved that angle A equals one half the central angle intercepting BC.

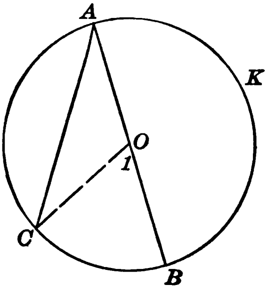


Fig. 73.

Draw OC. [Use § 102 to prove $\angle O_1 = 2\angle A$.]

2d. Let the center of the circle be within the angle.

3d. " " " " " " " outside " "

173. Cor. 1. *All the angles inscribed in the same segment are equal.*

174. Cor. 2. *An angle inscribed in a semicircle is a right angle.*

174a. (1) Construct the problems in § 88*e* and § 88*h*, using § 174.

(2) Construct Problem D (§ 57) from this figure.

[1st. Draw from P to any point, C, on AB.

2d. Draw the semicircle CDP.]

(3) To construct a right triangle with a given hypotenuse and a given altitude from the right angle to the hypotenuse.

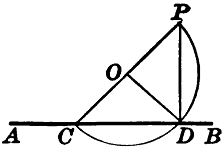


Fig. 74.

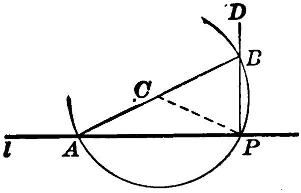


FIG. 75.

174b. Construct Problem C, § 56, from this figure.

l is the given line, P the given point.

[C is any point outside l.]

Exs. 1. The angle inscribed in a major segment is acute.

2. Find the locus of the vertex of the right angle (A) when the hypotenuse (BC) is fixed and the vertex (A) is movable.

3. Two chords (a, b) are perpendicular to a given chord (CD) at its extremities (C, D). Prove $a = b$.

PROPOSITION XXXVI. THEOREM.

175. *The angle between two chords intersecting within a circle is equal to one-half the sum of the central angles which intercept its arc and the arc of its vertical angle.*

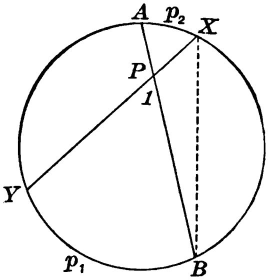


FIG. 76.

Let AB and XY be two chords intersecting at P, and intercepting arcs p_1, p_2.

It is to be proved that angle P_1 equals one-half the sum of the central angles intercepting p_1 and p_2.

PROPOSITION XXXVII. THEOREM.

176. *If two secants intersect outside a circle, the angle between them is equal to one-half the difference of the central angles intercepting the same two arcs.*

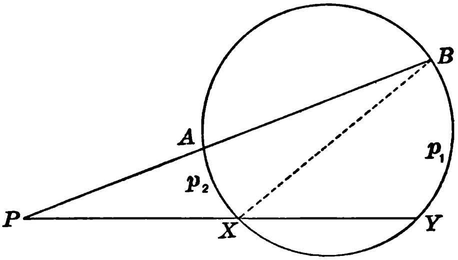


FIG. 77.

PROPOSITION XXXVIII. THEOREM.

177. *The angle between a tangent and a chord intersecting at the point of tangency is equal to one-half the central angle intercepting the same arc.* [$\angle T_1 = \angle O_1$.

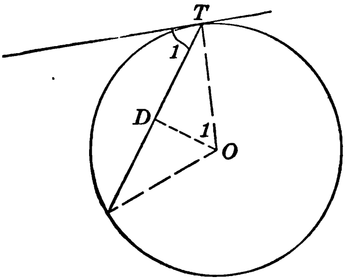


FIG. 78.

PROPOSITION XXXIX. THEOREM.

178. *The angle between a tangent and a secant which intersect outside a circle is equal to one-half the difference of the central angles intercepting the same two arcs.*

PROPOSITION XL. THEOREM.

179. *The angle between two intersecting tangents is equal to one-half the difference of the central angles intercepting the same two arcs.*

180. DEFS. A polygon is **inscribed** in a circle—or the circle is **circumscribed** about the polygon—when the vertices of the polygon lie in the circumference. The vertices are then said to be **concyclic.**

A polygon is **circumscribed** about a circle—or the circle is **inscribed** in the polygon—when the sides of the polygon are tangent to the circle.

EXS. 1. *The opposite angles of an inscribed quadrilateral are supplementary.*

2. *If the opposite angles of a quadrilateral are supplementary, the vertices are concyclic.*

PROPOSITION XLI. THEOREM.

181. *If two tangents to a circle intersect, the distances from their point of intersection to their points of contact are equal.*

182. COR. *The line joining the center of a circle to the point of intersection of two tangents bisects the angle between the tangents.* [Draw CT_1, CT_2.]

183. DEFS. Two **circles** are **tangent** to each other when they are tangent to the same straight line at the same point. They are tangent **externally** when each is external to the other. They are tangent **internally** when one is within the other.

EXS. 1. PROBLEM. At a given point on the circumference of a given circle to draw a circle with given radius tangent to the given circle.

2. *The line in* § 182 *bisects the central angle* T_1CT_2. *Make the theorem.*

3. The sum of two opposite sides of a circumscribed quadrilateral is equal to the sum of the other two sides.

4. What is the locus of the center of the circle tangent to two given lines?

5. The circle and ∠ A are fixed. As point P moves about the arc TT', show that the perimeter of the triangle ABC is constant. What does the perimeter equal?

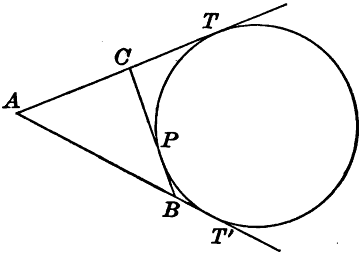


FIG. 79.

6. If lines are drawn from the center of a circle to the vertices of a circumscribed quadrilateral, two central angles opposite each other are supplementary.

184. THEOREM. *If two circles intersect, the line through their centers bisects their common chord.*

185. COR. *If two circles intersect, the line through their centers is perpendicular to their common chord.*

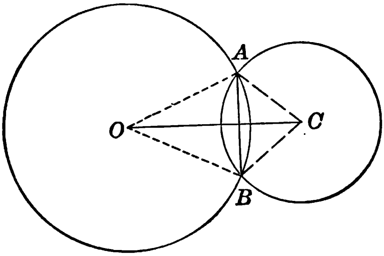


FIG. 80.

186. THEOREM. *If two circles are tangent, their centers and the point of tangency are in a straight line.*

[Draw OT, and prove that TO passes through C.]

187. Cor. *If two circles are tangent, the line through their centers is perpendicular to the tangent at their point of contact.*

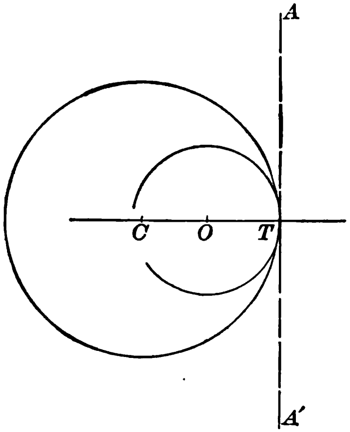


Fig. 81.

Exs. 1. What is the locus of the center of a circle passing through two given points?

2. What is the locus of the center of a circle tangent to a given line at a given point?

3. What is the locus of the center of a circle tangent to a given circle at a given point?

4. The radius of circle K is taken as the diameter of circle J, and the point of contact is T. Prove that a chord of K drawn from T is bisected by J.

188. Theorems. The following relations between two circles are sometimes assumed, but they are easily established and are left as exercises for the pupil.

(1) If two circles are wholly external to each other, the distance between their centers (or their center distance) is greater than the sum of their radii.

(2) If two circles are tangent externally, their center distance is equal to the sum of their radii.

(3) If two circles intersect, the center distance is less than the sum of their radii and greater than the difference of their radii.

(4) If two circles are tangent internally, their center distance is equal to the difference of their radii.

(5) If one circle is wholly within another, their center distance is less than the difference of their radii.

188*a*. The converse of each of the above theorems will be referred to this section with a corresponding number; thus 188*a* (1) is If the center distance of two circles is greater than the sum of their radii, etc.

189. Def. Two circles are **concentric** when their centers coincide.

Exs. 1. What is the locus of the center of a circle of constant radius always tangent to a given circle?

2. *a* (Fig. 82) is a rod of constant length sliding so that its extremities are always on the sides of the right angle *R*. Find the locus of the middle point of *a*.

3. A straight (l) is drawn through a fixed point (P) intersecting a given circle in two points (A, B). When l turns about P, find the locus of the middle point of AB.

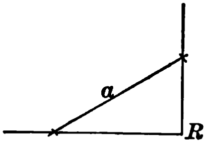


Fig. 82.

Proposition H. Problem.

190. *To draw a tangent to a given circle from a given point without the circle.*

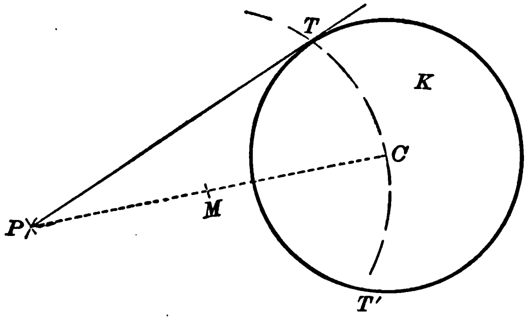


Fig. 83.

[Prove that TP is tangent by showing $\angle\ PTC$ is a $\underline{|R}$.]

Exercise. Solve § 190 by § 88*h*.

EXERCISES.

1. Through a given point within a circle construct a chord equal to a given line.

2. If two circles are tangent to a line at T, and if from any point (A) in the line a tangent (AK, AL) is drawn to each circle, these two tangents are equal.

3. Prove that TA bisects PQ.

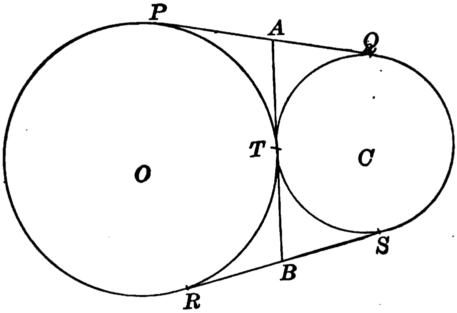


FIG. 84.

4. Prove that $PQ = RS$.

5. Prove that $AB = PQ$.

6. Prove that $\angle OAC$ is a right angle.

7. Prove $\angle PTQ$ a right angle.

NOTE. Constructions may sometimes be solved by discovering two loci the intersections of which give required points.

8. To construct a circle passing through a given point and tangent to a given line at a given point. [Page 66, Ex. 1, gives one locus, Ex. 2 gives another. Their intersection gives the required center.]

9. To construct a circle tangent to a given line at a given point and also tangent to another given line.

10. To construct a circle tangent to a given circle at a given point and also tangent to a given line.

11. To construct a circle tangent to a given line at a given point and also tangent to a given circle.

12. To construct a circle with a given radius—

    (1) tangent to two given lines;
    (2) tangent to a given line and to a given circle;
    (3) tangent to two given circles.

Proposition J. Problem.

191. *To circumscribe a circle about a given triangle.*

[Prove that S is the center and SA is the radius. Why do lines K and H intersect?]

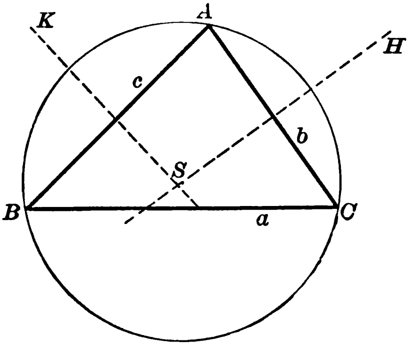


FIG. 85.

192. COR. 1. *To pass a circumference through three given points (not in a straight line).*

193. COR. 2. *To construct the center of a given arc.*

NOTE. The required circle of § 191 is called the circumcircle of the triangle ABC. S is called the circumcenter. SA is a circumradius.

Proposition K. Problem.

194. *To inscribe a circle in a given triangle.*

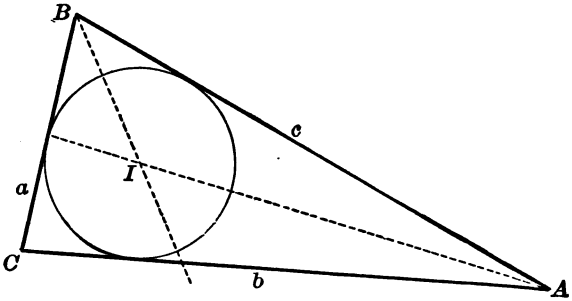


FIG. 86.

195. Cor. *To draw a circle tangent to three given lines.* [When is it impossible?]

196. Sch. A circle may be drawn externally tangent to any side of a triangle, and also tangent to the two other sides produced. Such a circle is *escribed* to the triangle.

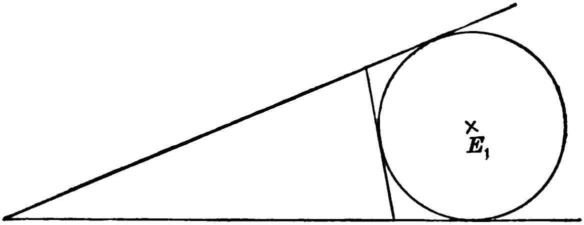


Fig. 87.

Note. The required circle of § 194 is called the incircle of the triangle; its center is the incenter, its radius the inradius of the triangle. The escribed circles of § 196 are called ecircles, etc.

Exercises. Denote the half-perimeter of the triangle ABC by s.

1. The distance of the vertex A from a point of contact of the incircle is $s-a$, and similarly for the other vertices.

2. Prove an analogous relation (to the one just given) for the points of contact of the ecircles.

3. To construct a triangle when the perimeter and the angles are given.

PROPOSITION M. PROBLEM.

197. *To construct on a given chord a segment containing a given angle as an inscribed angle.*

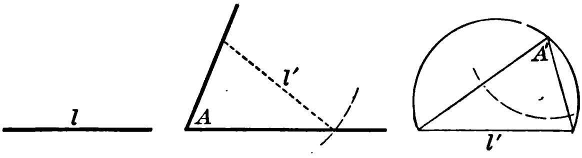


FIG. 88.

EXS. 1. To construct a triangle when the base, the altitude and the vertical angle are given.

2. To construct a triangle when two sides and the angle opposite one side are given.

3. To construct a triangle when the base, the vertical angle and the median are given.

4. What is the locus of the vertex of a triangle when the base and vertical angle are given ?

PROPOSITION N. PROBLEM.

198. *To draw a common tangent to two given circles.*

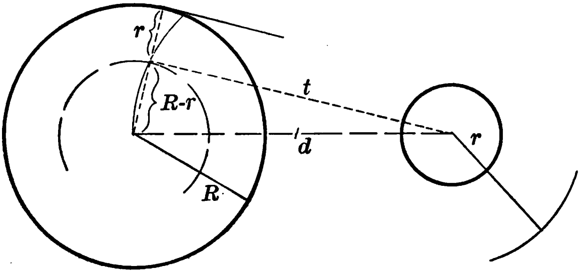


FIG. 89.

How many tangents may be drawn to two given circles ?

NOTE. The tangent suggested in the figure for § 198 is called the *direct* common tangent. Another tangent can be drawn which crosses the line of centers between the circles. This is called the *transverse* tangent.

199. Let us assume the fixed indefinite line l, the fixed point P, the fixed line PA.

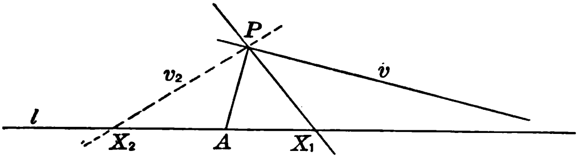


FIG. 90.

Let v be an indefinite line which is to be turned about P as a pivot, and let X denote the point where v intersects l. As v turns about P four parts of the triangle PAX vary, while the two parts side PA and angle A are constant.

(1) These varying quantities are *continuous* variables. AX, for example, while changing from one inch to two inches, passes through all values between one and two inches.*

Notice in the same way the continuity of the other variables.

(2) Suppose v is turned "clockwise" through the position PA to the position v_2. The line AX_2 is now read from right to left, and this reversal of its direction as compared with AX_1 is denoted by the positive and negative signs of algebra. If AX_1 is $+$, then AX_2 is $-$.

Notice a similar relation in the angle APX.

(3) As v is turned clockwise through A, AX continually decreases till it becomes zero, and then reappears as a negative line. This idea is stated: As v passes through A, AX passes through zero.

But if v is turned "counter-clockwise" from the position PX_1, AX_1 increases continuously until v is parallel to l. When this occurs there is no point of intersection X; and as there is no boundary at the right or left of A, AX is said to be *infinite*, and X is said to be *at infinity*. If now v is turned beyond the parallel position, X reappears on the left of A—that is, AX is negative. This idea is stated: As v passes through parallelism to l, AX passes through infinity.

---

* As an illustration of a variable *not* continuous, take a triangular card and clip the corners repeatedly. The figure may be made to have any number of sides,—and the *number* of sides is a variable. But it (the number) is not continuous; there is no $4\frac{1}{2}$-sided polygon.

(4) As v turns about P the line PX is a continuous variable, but it does not pass through 0; neither is it convenient to regard it as changing sign. It is therefore considered either signless or positive. Suppose v is turned counter-clockwise from v_2 to v_1, PX continuously decreases, reaches a shortest length and then increases. Its shortest length is called a *minimum.*

DEFS. If a continuous variable reaches a greatest value and then decreases, that greatest value is a **maximum** (*max.*).

If a continuous variable reaches a least value and then increases, that least value is a **minimum** (*min.*).

EXERCISES. 1. A revolving chord passes through a fixed point in a given circle.

When is the chord a max.?

When is the chord a min.?

Suppose the point is in the circumference.

2. From a fixed point P in a circle a variable line PX is drawn to the circumference. When is PX a max.? When a min.? Suppose P is in the circumference. Suppose P outside the circle.

3. A triangle ABC has a fixed base AB and a constant altitude CD. When is the angle C a max.?

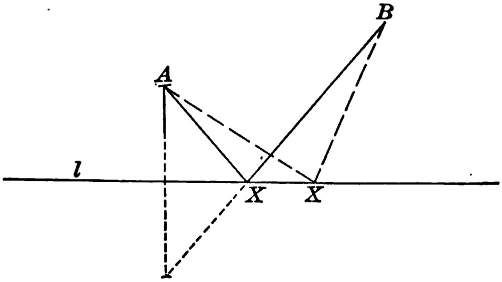


FIG. 91.

4. The fixed points A, B are on the same side of the fixed indefinite line l. X is a movable point in l. When is the length AXB a min.?

5. Two fixed points, A and B, are on opposite sides of the fixed indefinite line l. X is a movable point in l. When is the difference of AX and BX a max.?

200. Let AX (I) and BY (II) be two chords (or secants) intersecting at Q. Let the arrow indicate positive direction about the circumference. Then arcs AB and XY are positive.

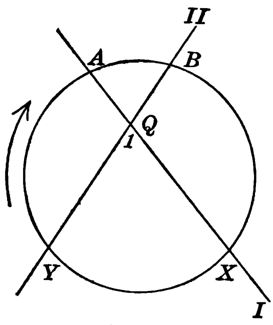


FIG. 92.

(1) As I is turned clockwise about X as a pivot, AB decreases continuously. When Q is on the circumference, AB is 0. When Q is outside the circumference, A is at the right of B, and the minor arc AB is negative. Follow the position of Q and the value of the angle Q_1 until I passes parallelism to II.

(2) As I is turned about X the point A may be made to approach X until I is a tangent; then A is at X. For this reason a tangent is regarded as a special case of a secant, and is sometimes defined as a secant whose points of intersection coincide. The point of tangency is then called a double point.

(3) By choosing other points (as B or Q) for pivots the secants may be brought into all the relations treated in §§ 172, 175–179; and the student will see that by properly regarding the signs of the intercepted arcs those sections may be generally stated in this one theorem:

The angle between two secants to a circle is equal to one-half the algebraic sum of the central angles intercepting the same two arcs.

(See also §§ 146, 147, as related to §§ 168, 169).

200*a*. If two circles intersect, by moving one or both circles the two points of intersection may be made to coincide. § 186 is then included in § 184; and § 187 is included in § 185.

EXERCISES.

1. In the triangle ABC the bisector of angle A and the bisector of the exterior angle at B form an angle equal to one-half angle C.

2. The lines from a vertex of a parallelogram to the mid points of the opposite sides trisect a diagonal.

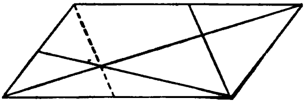


FIG. 93.

3. The number of diagonals in a polygon of n sides is $\frac{n(n-3)}{2}$. [How many diagonals can be drawn from one vertex?]

4. Construct a quadrilateral when five elements are given. State the problems.

DEF. The triangle formed by joining the feet of the altitudes of a triangle is called the **pedal** triangle.

5. In a right triangle the diameter of the incircle is equal to the difference between the sum of the sides and the hypotenuse.

6. The altitudes of a triangle bisect the angles of the pedal triangle. [Take the sides of the given triangle as diameters of circles.]

7. To construct a triangle when the feet of the altitudes are given.

8. Two circles are tangent at T. Draw ATX, BTY. Prove AB parallel to XY. [Draw a tangent line at T.]

9. Two circles (I, II) intersect at X, Y. From X draw a diameter in each circle (XA, XB). Prove A, Y, B in a straight line.

10. If the opposite sides of an inscribed quadrilateral be produced till they intersect, the bisectors of the angles at the points of intersection are perpendicular.

11. Draw a line equally distant from three given points.

12. Through a given point draw a line forming an isosceles triangle with two given lines.

13. ABC is an equilateral triangle, and P is any point in the arc BC of the circumscribed circle. Prove that $PA = PB + PC$.

14. If a secant is drawn through each point of intersection of two intersecting circles, the chords joining the extremities of the secants are parallel.

15. On a given triangle construct circles having the vertices as centers, so that the circles shall be externally tangent to each other. Extend the problem.

16. Find a point (O) within a triangle (ABC), such that the three lines (OA, etc.) drawn from the point to the vertices shall make equal angles at the point. (When impossible?)

17. In the triangle ABC a point is taken on each side and circumcircles are drawn to the triangles BXZ, CXY intersecting in O. To prove that the circumcircle to AYZ passes through O. [Angles B and O_2 are suppl., C and O_3 are suppl. ∴ A and O_1 are suppl. Therefore A, Y, O, Z are concyclic.]

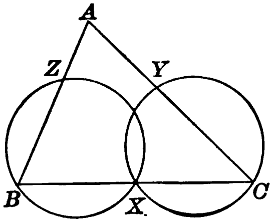


FIG. 94.

18. Prove ex. 17 when one or more of the points X, Y, Z are taken on the sides produced.

Prove ex. 17 when the three points are in a straight line.

(See Clifford's "The Common Sense of the Exact Sciences," pp. 78–81.)

19. P is any point in the circumference of the circumcircle to the triangle ABC, and PX, PY, PZ are the perpendiculars to the sides of the triangle from P. Prove X, Y, Z in a straight line.

[Angle $PXZ = PCA =$ suppl. PXY.]

The line XYZ is called the *Simson* line of the triangle.

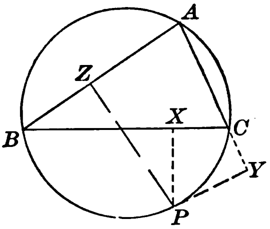


FIG. 95.

20. Construct a right triangle given—

(1) the median and the altitude to the hypotenuse;
(2) the sum of a and b, and A;
(3) the difference of a and b, and A;
(4) the perimeter and a;
(5) the perimeter and A;
(6) r (the inradius) and a;
(7) r and A.

21. Construct a triangle given—

(1) the incircle and the angles;
(2) the circumcircle and the angles;
(3) the three medians.

22. Construct externally tangent to a given circle three circles equal to each other and each externally tangent to the others.

23. Construct externally tangent to a given circle six circles equal to each other and each tangent to two required circles.

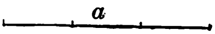
BOOK II.

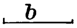
PROPORTION; SIMILAR FIGURES; REGULAR FIGURES.

PRELIMINARY.

201. The magnitudes of plane geometry are lines, angles, and areas. Straight lines are the simplest illustration of magnitudes, but the relations which are considered here may be applied later to angles, arcs, and areas.

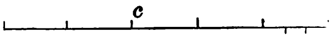
202. When we attempt to measure one magnitude by another three cases arise.

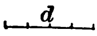
(1) In the first place, the first magnitude may contain the second a whole number of times. The second is called the unit and the first is called a multiple of the unit. Thus 

contains 

three times, or $a = 3b$. 3 is the measure of a when b is the unit.

(2) In the second place, the first magnitude may not contain the second a whole number of times, but both the first and second magnitudes may contain a third magnitude a whole number of times.

Thus 

and 

each contain

a whole number of times.

In this illustration—

(1) $c = 15s,$

(2) $d = 4s;$

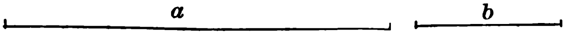
from (1), $4c = 4 \times 15s = 15 \times 4s;$

from (2), $= 15d;$

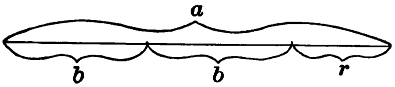
$\therefore\ c = \frac{15}{4}d.$

From this last equation we say that c *contains* d $\frac{15}{4}$ times, and $\frac{15}{4}$ is regarded as a number; that is, $\frac{15}{4}$ is the number of times that c contains d,—or $\frac{15}{4}$ is the measure of c when d is the unit.

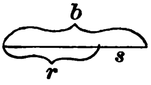
A common measure of two magnitudes is found by a process similar to the arithmetic process of finding the greatest common divisor of two numbers.. Thus take the magnitudes

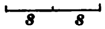


Apply the smaller to the larger; b is contained in a twice with a remainder r.



Apply r (which is of course less than b) to b; r is contained in b once with a remainder s.



Apply s to r; s is contained in r just  twice.

Then $r = 2s$; $b = r + s = 3s$; $a = 2b + r = 8s$.

That is, b and a each contain s, or s is a common measure of a and b.

If the above example is carefully examined, the following statements will be evident:

If a and b have a common measure, m, the "remainder" r will exactly contain m; and if b and r have a common measure, m, the next remainder, s (if there is a remainder), will exactly contain m; and so on.

In short, if a and b have a common measure, every remainder contains it, and no remainder (except zero) can be smaller than it.

But the remainders become smaller at every step, and always smaller by some multiple of m. Therefore by dividing (or "applying") some number of times we shall reach a remainder zero.

(3) In the third place, it may occur that the two magnitudes have no common measure.

Suppose we attempt to find the common measure of the side and diagonal of a square (Fig. 96). Call a side $= s$, and the diagonal $DB = h$. Apply s to h; it is contained once (§ 75) and not twice (§ 89a). Let $DP = s$; draw $PQ \perp DB$. Then $PB = PQ = QC$ (page 50, ex. 4). Call $PB = r$.

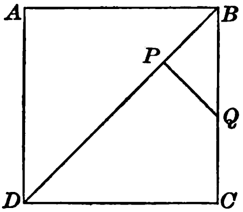


FIG. 96.

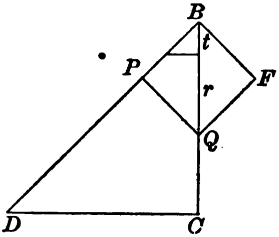


FIG. 97.

Apply r to s (Fig. 97). It is contained twice with a remainder t.

We are now seeking the common measure of r and BQ, that is, of a side and diagonal of the square $BPQF$; and at the next step we shall be seeking the common measure of a side and diagonal of a still smaller square.

Evidently at every step of the work the side of the square we are then examining will be contained in its diagonal more than once and less than twice. Therefore the process is endless; that is, no common measure will occur. In this case the two magnitudes are "incommensurable."

203. When the two magnitudes are incommensurable the common measure of the first and of two other magnitudes—one greater and the other less than the second given magnitude—can always be found. For one magnitude—the first—may be divided into a certain number (n) of equal parts. Suppose the second magnitude contains x

of these parts, with a remainder smaller than one part. Then $(x+1)$ parts form a magnitude larger than the second given magnitude, and x parts form a magnitude smaller than the second given magnitude.

Thus in the square above, if the unit s be divided into 1024 equal parts (§ 129), 1448 of these parts will almost extend from D to B. Suppose they extend to B', leaving a remainder, $B'B$, smaller than one of the parts. Then 1449 parts would extend beyond B', say to G. The numerical measure of DB' is $\frac{1448}{1024}$ when s is the unit, and the measure of DG is $\frac{1449}{1024}$ when s is the unit.

Observe these points:

(1) *It is assumed* that DB has a *numerical measure* when s is the unit.

(2) It is inferred that the numerical measure of DB falls between $\frac{1448}{1024}$ and $\frac{1449}{1024}$ when s is the unit.

204. Now by dividing s into still smaller parts, two lengths—one less and one greater than DB—can be found which shall more nearly coincide with DB. Thus if s is divided into 8192 equal parts, 11,589 of these parts will extend, say, to B'', leaving a remainder less than one of these parts. Then obviously the numerical measure of DB falls between $\frac{11,589}{8192}$ and $\frac{11,590}{8192}$.

In this way we may find the numerical measures of a set of magnitudes which *approach* DB as near as we please without precisely coinciding with it. Therefore—

(3) It is inferred that we can *approximate* the true numerical value of DB when s is the unit.

205. Def. **A number** is

(*a*) an integer or } **arithmetic or "rational"**
(*b*) a fraction, } **numbers,**
or (*c*) an "irrational"

lying between two rational numbers which are expressible in arithmetic.

EXERCISES.

In finding the approximate value in these exercises the two approximations between which the true value lies should be found.

1. Divide the diagonal of a square into 10 equal parts and find the approximate values of one side. (One side is 0.7 + when the diagonal is the unit.)

2. Divide the side of a square into 10 equal parts and find the approximate value of the diagonal.

3. The base of a right triangle is *one* and the hypotenuse is *two* units. How long is the altitude (1) when the unit is divided into thirds? (2) When the unit is divided into fourths?

4. The base of a right triangle is 1 and its altitude is 2. Find the hypotenuse to fifths.

5. The base of a right triangle is 4 and its altitude is 3. Find the hypotenuse within a unit.

6. A chord of a circle equals one third the radius, and the subtended arc equals a. Find the length of the semicircumference when a is the unit.

7. In exercise 6 substitute *one-half* for one-third.

8. In exercise 6 substitute *one-sixth* for one-third.

9. Find the length of a quadrant when the unit is one-eighth of the arc subtended by a radius.

10. Take one-eighth of the quadrant as a unit and find the length of the arc subtended by a radius. Observe that the results of exs. 9 and 10 should be consistent.

NOTE. Many of the irrational numbers which occur commonly in geometry are written in arithmetic; when it is said that the diagonal of a square whose side is unity is not expressible in arithmetic it is meant that no finite set of figures can express the true value. We can *indicate* the true value by saying diagonal $= \sqrt{2}$; and we can *approximate* the true value.

The same rules of operation that control rational numbers are applied to irrationals.

THEORY OF PROPORTION.

206. Defs. The **ratio** of two magnitudes of the same kind (or of two numbers) is their relation with respect to size.

Ratio is denoted by a colon. ($a : b$ is read, "the ratio of a to b.)

The **terms** of a ratio are the magnitudes (or numbers) compared. The first term is called the **antecedent**, and the second term is called the **consequent.**

Two magnitudes (or two numbers) are **commensurable** when they have a common measure; they are **incommensurable** when they have no common measure.

A **commensurable ratio** is a ratio of two commensurable magnitudes (or numbers).

An **incommensurable ratio** is a ratio of two incommensurable magnitudes (or numbers).

207. Def. The numerical value (or **the value**) of a ratio of *two magnitudes* is the number of times the first magnitude contains the second.

208. Cor. *If the value of the ratio of the first to the second magnitude is equal to n, then the first is equal to n times the second.*

(If $A : B = n$, then $A = B . n$.)

209. Def. The **value** of a *numerical* ratio is the quotient arising from dividing the first number by the second.

210. Sch. Ratio is a relation which includes—

(1) The *measuring* of one magnitude by another.

(2) The *dividing* of one number by another.

211. THEOREM. *The ratio of two magnitudes is equal to the ratio of their numerical measures when referred to the same unit.*

Let the magnitudes be A, B, and let their respective measures referred to u be h, k.

Then $$A = h \, . \, u, \qquad B = k \, . \, u.$$
$$A : B = h \, . \, u : k \, . \, u = h : k.$$

212. DEFS. A **proportion** is a statement that two ratios are equal. $a : b = x : y$ is read, "the ratio of a to b equals the ratio of x to y"; also read, "a is to b as x is to y."

The equality sign is sometimes replaced by four dots (: :).

The **extremes** are the first and fourth terms; the **means** are the second and third terms.

213. SCH. The terms of a proportion may be the magnitudes of geometry, or they may be numbers, or one ratio may be between two magnitudes (of the same kind), and the other ratio may be between numbers.

214. DEF. A **fourth proportional** to three quantities is the fourth term of a proportion in which the three quantities are the first three terms in the order given. Thus Y is the fourth proportional to A, B, X if $A : B = X : Y$.

215. DEF. A **mean proportional** between two quantities is the second and also the third term of a proportion in which the two quantities are extremes. Thus X is the mean proportional between A and B if $A : X = X : B$.

216. DEF. A **third proportional** to two quantities is the fourth term of a proportion in which the second quantity is a mean proportional between the others. Thus X is a third proportional to A and B if $A : B = B : X$.

217. THEOREM. *If four magnitudes are in proportion, either ratio may be replaced by an equal numerical ratio.*

[§ 211.

218. COR. *A proportion may be replaced by an equation between two fractions whose terms are the numerical measures of the respective terms of the proportion.*

219. THEOREM. *If four numbers are in proportion, either ratio may be replaced by an equal ratio of magnitudes.*

220. COR. *An equation between two fractions may be replaced by a proportion whose terms* (*of the same kind in each ratio*) *are measured numerically by the respective terms of the fractions.*

221. SCH. It follows from § 219 that any transformation of an algebraic proportion demonstrates a like transformation of a proportion of magnitudes, *provided* the resulting proportion has an interpretation in geometry.

222. THEOREMS IN PROPORTION. I. *If four magnitudes of the same kind are in proportion, they are in proportion by* **Alternation**;

that is, the first is to the third as the second is to the fourth.

Let A, B, C, D be four magnitudes of the same kind in proportion, and let a, b, c, d be their respective numerical measures to the same unit.

It is to be proved that

$$A : C = B : D.$$

$$A : B = C : D;$$

then $$\frac{a}{b} = \frac{c}{d}. \qquad [?$$

$$\therefore \quad \frac{a}{c} = \frac{b}{d}. \qquad [\text{Mult. by } \frac{b}{c}.$$

Then $$A : C = B : D. \qquad [?$$

In I it is necessary that *all four* magnitudes be of the same kind; while in II–V it is only necessary that the magnitudes *in each ratio* be of the same kind. Thus II is stated, "If four magnitudes are in proportion, they are in proportion by *inversion*; that is," etc.

The transformations II–V, with their definitions, are indicated in the following algebraic proportions:

If $a : b = c : d,$ then

II. $b : a = d : c;$ by **Inversion.**

$$\left.\begin{array}{ll}\text{III. (1)} & (a + b) : a = (c + d) : c, \text{ or} \\ \text{(2)} & (a + b) : b = (c + d) : d;\end{array}\right\} \text{by } \textbf{Composition.}$$

$$\left.\begin{array}{ll}\text{IV. (1)} & (a - b) : a = (c - d) : c, \text{ or} \\ \text{(2)} & (a - b) : b = (c - d) : d;\end{array}\right\} \text{by } \textbf{Division.}$$

V. $(a + b) : (a - b) = (c + d) : (c - d)$ by **Composition and Division.**

VI. In a series of equal ratios of magnitudes of the same kind the sum of any number of antecedents is to the sum of the corresponding consequents as any antecedent is to its consequent.

Let $A : B = C : D = E : F = G : H.$

It is to be proved that

$$(A + C + G) : (B + D + H) = C : D.$$

Let a, b, c, d, e, f, g, h be the respective measures of $A, B, C, D, E, F, G, H.$

Also let the numerical value of any ratio be denoted by $r.$

$$\text{Then} \quad \left.\begin{array}{l} A : B = r, \\ C : D = r, \\ G : H = r, \end{array}\right\} \text{ or } \left\{\begin{array}{l} A = Br, \\ C = Dr, \\ G = Hr. \end{array}\right. \qquad [?$$

$$\text{Adding,} \quad \begin{aligned} A + C + G &= Br + Dr + Hr \\ &= (B + D + H)r. \end{aligned}$$

$$\text{Then} \quad \begin{aligned} (A + C + G) : (B + D + H) &= r \qquad [\S\ 210\ (1). \\ &= C : D. \end{aligned}$$

VII. If two proportions have three terms of one equal respectively to three terms of the other, the fourth terms are equal.

Let $a : b = c : f,$

and $a : b = c : f'.$

It is to be proved that $f = \;'.$

VIII. If two proportions have the extremes of one equal to the extremes of the other, the mean proportional of the first is equal to the mean proportional of the second.

Let $a : x = x : b,$

and $a : x' = x' : b.$

It is to be proved that $x = x'.$

NOTES. 1. When three or more ratios are equal the proportion is often written in a "continued" form, thus: $a : b :: c : d :: e : f :: g : h$; and the same proportion is denoted by the following form:

$$a : c : e : g :: b : d : f : h.$$

2. In the following form, $a : b :: b : c :: c : d,$ b and c are two mean proportionals between a and d.

LIMITS.

223. DEF. A **variable quantity** (or a variable) is a quantity whose value is supposed to change.

224. DEF. A **constant** is a quantity whose value is fixed.

225. DEF. An **independent variable** is a quantity whose value is changed at pleasure.

226. DEF. A **dependent variable** is a quantity whose value is changed by changing the value of another variable on which the first depends.

If a line is divided into four, then into eight, then into sixteen equal parts, the *number* of parts into which the line is divided is an independent variable, and the length of each part is a dependent variable.

227. DEF. The **limit** of a dependent variable is a constant which the dependent variable may be made to approach as near as we please, but with which the variable can never exactly coincide.

The incommensurable ratio $h:s$ on page 80 is a limit which the commensurable ratios approach. The independent variable is the number of parts into which the side s is divided.

227*a*. If the difference between the variable and a constant can be made less than any finite quantity, but cannot be made equal to zero, then the constant is the limit of the variable. This is equivalent to the definition of the limit of a variable just given.

228. SCH. The independent variable is changed (often increased) in some methodical way so as to exhibit the approach of the dependent variable to its limit.

229. THEOREM. *If two variables, dependent on the same variable, are always equal, and if each approaches a limit, then the limits are equal.*

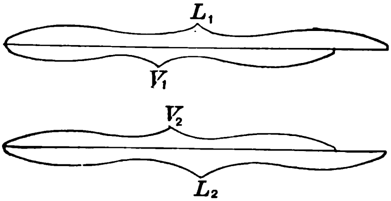


FIG. 98.

Let V_1 and V_2 be two variables depending on the same variable, and let L_1 and L_2 be their respective limits, and let V_1 always equal V_2.

It is to be proved that L_1 equals L_2.

1. Suppose V_1 and V_2 increasing (i.e., always less than their limits).

If L_1 does not equal L_2, one of them (say L_1) is the greater.

Then V_1 may be made greater than L_2. [§ 227.

But V_2 is always less than L_2. [?

Then V_1 and V_2 are unequal.

But this is impossible.

Therefore L_1 cannot be greater than L_2.

In the same way it can be shown that L_1 cannot be less than L_2.

Therefore L_1 equals L_2.

2. Suppose V_1 and V_2 decreasing (i.e., always greater than their limits). [Use the same method of proof.]

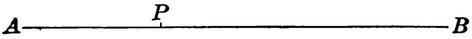
230. SCH. In the cases which occur in elementary geometry the variable is either always less, or else always greater, than its limit.

In certain series in algebra the variable is alternately greater and less than its limit. This case need not be considered here.

PROPORTIONAL LINES AND SIMILAR FIGURES.

231. DEFS. The **segments** of a line are the parts of the line from any point in the line to the extremities.

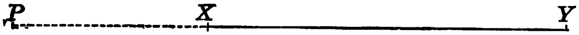
Thus the line AB is divided at P into the segments PA, PB.



The line is divided **internally** when the point of section is between the extremities of the line.

The line is divided **externally** when the point of section is on the line produced.

Thus XY is divided externally at P, and PX, PY are external segments.



232. DEF. Two lines are **divided proportionally** when the ratio of the segments of one line is equal to the ratio of the corresponding segments of the other line.

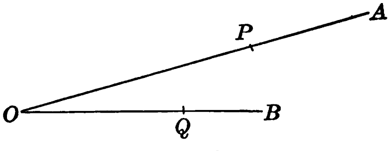


FIG. 99.

Thus OA and OB are divided proportionally at P, Q if $PO : PA$ equals $QO : QB$. The lines must be divided either both internally or both externally.

By § 222, III, it follows that $OA : OP :: OB : OQ$, and this form is included in the definition.

PROPOSITION I. THEOREM.

233. *A line parallel to one side of a triangle divides the other two sides proportionally.*

Let ABC be a triangle, and let HK be parallel to BC.

It is to be proved that $HA : HB :: KA : KC$.

CASE I. When the segments of one side are commensurable.

Let AH and HB be commensurable.

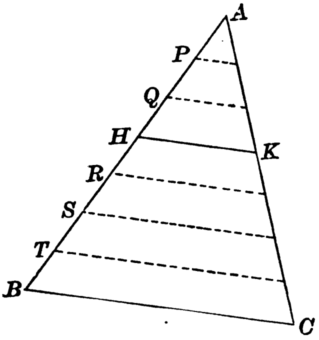


FIG. 100.

Find the common measure of HA and HB. [Page 78.

Call this measure m.

Apply m as a measure to HA and HB.

It will be contained in HA some integral number of times (say f times), and in HB g times.

Then $HA : HB = f : g$.

Through the points of division on HA and HB draw parallels to BC, cutting AC in T', S', etc.

Then $CT' = T'S' = S'R'$, etc. [?

That is, $KA = f . AP'$, and $KC = g . AP'$; or

$KA : KC = f : g$. [§ 211.

$\therefore$ $HA : HB :: KA : KC$.

Case II. When the segments of one side are incommensurable.

Let AH and HB be incommensurable.

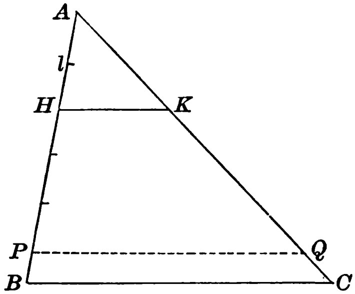


Fig. 101.

Divide HA into any number of equal parts.

Apply one of these parts, l, to HB as a measure.

Some number of these parts will extend to P, and there will be a remainder, r, smaller than l. [?

Draw PQ parallel to HK.

Then $HA : HP :: KA : KQ$. [?

Divide HA into so large a number of equal parts that one part (say l') shall be smaller than r.

Apply l' to HB as a measure.

The new remainder, r', will be smaller than r.

By increasing indefinitely the *number* of equal parts of HA, one of the parts may be made less than any finite line.

$\therefore$ the remainder PB, and also QC, may be made less than any finite line.

$\therefore$ the variable HP is made to approach HB as a limit, and the variable KQ is made to approach KC as a limit.

$\therefore$ the variable $HA : HP$ approaches $HA : HB$ as a limit, and the variable $KA : KQ$ approaches $KA : KC$ as a limit.

But the variable $HA : HP$ is always equal to the variable $KA : KQ$.

$\therefore$ the limit $HA : HB$ is equal to the limit $KA : KC$. That is, $HA : HB :: KA : KC$. Q. E. D.

NOTE.—The student should see clearly in what way QC may be made less than any finite length.

For any length of QC (say b) there is a corresponding length of PB (say a). If l is made shorter than a, a point of division falls within a, and QC is shorter than b.

234. SCH. The proportion $HA : HB :: KA : KC$ may be written (§ 222, I and III):

(1) $AB : AH :: AC : AK$,

(2) $AB : HB :: AC : KC$,

(3) $AB : AC :: AH : AK :: HB : KC$,

and these are all implied in the statement of the theorem § 233.

EXS. 1. Make P fall outside AB.

2. Draw HK below BC.

3. Also above A.

4. What does the demonstrated proportion become "by composition" (§ 222, III)?

5. What by alternation?

6. If two parallels are cut by three transversals from a point, what is the proportion?

7. If three parallels are cut by two transversals from a point, what is the proportion?

8. Show how to divide a given line AB in the ratio of two given lines h and k.

9. Divide a given line externally in a given ratio. (See ex. 3.)

10. Divide a line both internally and externally in a given ratio.

11. In Fig. 101 draw KJ parallel to AB; then prove

$$AC : AK :: BC :: HK.$$

PROPOSITION II. THEOREM.

235. *If a line divides two sides of a triangle proportionally, it is parallel to the third side.*

In the triangle ABC let PR divide the sides AB, AC proportionally.

It is to be proved that PR is parallel to BC.

Through R draw RX parallel to BC.

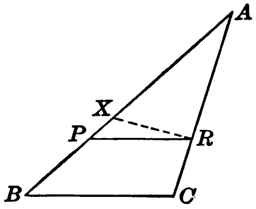


FIG. 102.

Then $AC : AR :: AB : AX$. [?

But $AC : AR :: AB : AP$. [?

$\therefore AP = AX$. [?

$\therefore PR$ coincides with RX [?
and is parallel to BC.

236. DEFS. Two polygons are **mutually equiangular** when the angles of one are equal respectively to the angles of the other.

Two polygons are **mutually equilateral** when the sides of one are equal respectively to the sides of the other.

237. DEF. Two polygons are **similar** when they are mutually equiangular and have their homologous sides proportional.

238. DEF. The **ratio of similitude** of two similar polygons is the ratio of any two homologous sides.

Ex. Prove the triangle APR similar to the triangle ABC.

PROPOSITION III. THEOREM.

239. *If two triangles are mutually equiangular, they are similar.*

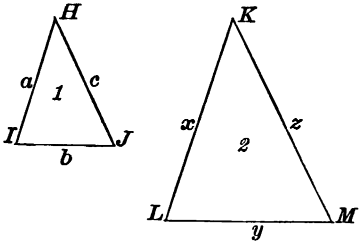


FIG. 103.

In the triangles HIJ, KLM, let $\angle H = \angle K$, $\angle I = \angle L$, $\angle J = \angle M$.

It is to be proved that *1* and *2* are similar.

From K on x lay off $KF = a$, on z lay off $KG = c$.

Draw FG.

Then prove $a : x : : c : z$. [§§ 44, 96, 233.]

Also from M lay off equals of b and c, and prove

$$b : y : : c : z.$$

$$\therefore a : x : : b : y : : c : z.$$

$\therefore$ the triangles *1* and *2* are similar.

240. COR. 1. *If two triangles have an angle of one equal to an angle of the other, and the sides including these angles proportional, the triangles are similar.*

241. COR. 2. *If two triangles have two angles of one equal respectively to two angles of the other, the triangles are similar.*

242. COR. 3. *If two right triangles have an acute angle of one equal to an acute angle of the other, the triangles are similar.*

EXERCISE. *A line parallel to the base of a triangle forms a triangle similar to the given triangle.*

PROPOSITION IV. THEOREM.

243. *If two triangles have three sides of one proportional to the three sides of the other, the triangles are similar.*

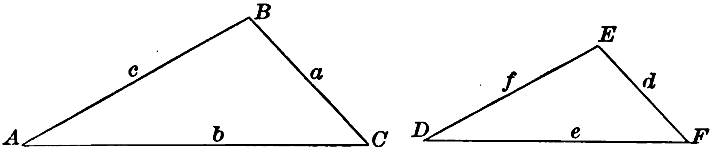


FIG. 104.

In the triangles ABC, DEF, let

$$a : d :: b : e :: c : f.$$

It is to be proved that the triangles are similar.

On b lay off $AH = e$; on c lay off $AK = f$.

Draw HK $(= x)$.

Then $b : e :: a : x$. [?

But $b : e :: a : d$.

$\therefore x = d$. [?

$\therefore$ triangle $DFE =$ triangle AHK. [?

But $\triangle AHK$ is similar to $\triangle ACB$; [?

$\therefore$ $\triangle ABC$ is similar to $\triangle DEF$.

244. COR. *The homologous altitudes of two similar triangles are in the ratio of similitude.*

NOTE. In general it can be shown that in similar triangles any homologous lines are in the ratio of similitude.

EXERCISE. Prove that in similar triangles (1) the homologous medians, (2) the inradii, (3) the circumradii are in the ratio of similitude.

245. THEOREM. *The perpendicular from the vertex of the right angle to the hypotenuse of a right triangle forms two similar triangles, each of which is similar to the given triangle.*

246. Cor. 1. *The altitude to the hypotenuse of a right triangle is a mean proportional between the segments of the hypotenuse.*

247. Cor. 2. *Either side of a right triangle is a mean proportional between the hypotenuse and the adjacent segment formed by the altitude to the hypotenuse.*

Exercises.

1. If transversals are drawn from a point to two parallels, the corresponding segments of the parallels are proportional.

2. If corresponding segments of two parallels are proportional, the transversals are concurrent.

3. The locus of the middle point of the line joining two sides of a triangle and parallel to the base is the median.

4. Two altitudes of a triangle are inversely proportional to their respective bases.

5. The segments of the diagonals of a trapezoid are proportional.

6. If through the intersection of the diagonals of a trapezoid a line is drawn parallel to the bases, this line is bisected at the point of intersection.

7. If two intersecting transversals are cut by a set of parallels, the corresponding segments of the transversals are proportional.

8. Is the converse of ex. 7 true ?

9. Prove § 134 using similar triangles.

10. The common tangent of two externally tangent circles is a mean proportional between their diameters.

11. If through the point of contact of two tangent circles secants are drawn, the segments are proportional.

12. Two parallel tangents are drawn to a circle, and a third tangent (at T) cuts them at A, B. Prove the radius of the circle a mean proportional between AT, BT.

13. $ABCD$ is any trapezoid, and HK is parallel to the base. Prove $HI = JK$.

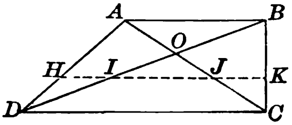


Fig. 105.

14. *If two triangles have the sides of one parallel respectively to the sides of the other, the triangles are similar.*

15. *If two triangles have the sides of one perpendicular respectively to the sides of the other, the triangles are similar.*

Proposition V. Theorem.

248. *The bisector of the vertical angle of a triangle divides the base into segments proportional to the sides.*

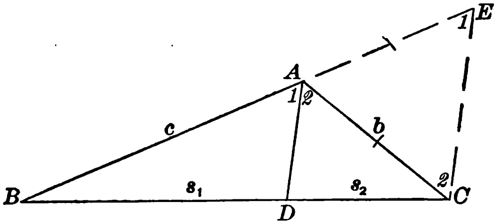


Fig. 106.

The triangle is ABC; AD is the bisector.

Prove ACE an isosceles triangle.

Exercises. 1. What is the converse?

2. Suppose the exterior $\angle\ CAE$ is bisected. Make the theorem.

3. If $BA = CA$, what is true? How about the bisector of the exterior angle?

248*a*. Cor. *The bisector of the exterior angle at the vertex of a triangle divides the base, etc.*

249. Def. The segments of two intersecting lines are **reciprocally proportional** when a segment of the first line is to a segment of the second as the remaining segment of the second is to the remaining segment of the first.

249*a*. Def. Two lines are **reciprocally proportional** to two other lines when the ratio of the first pair is equal to the inverted ratio of the second pair.

PROPOSITION VI. THEOREM.

250. *If two chords intersect within a circle, the segments of the chords are reciprocally proportional.*

EXERCISE. If chord *I* is a diameter, and *II* is perpendicular to *I*, what is true?

251. COR. *The perpendicular to a diameter from any point in a circumference is a mean proportional between the segments of the diameter.*

PROPOSITION VII. THEOREM.

252. *If two secants intersect outside a circle, the whole secants and their external segments are reciprocally proportional.*

253. COR. *If a tangent and a secant intersect, the tangent is a mean proportional between the whole secant and its external segment.*

EXERCISES. 1. The tangents to two intersecting circles from any point in their common chord (produced) are equal.

2. State and prove the converse of Prop. VI.

3. State the §§ 250, 252, 253 in one general theorem.

4. Use the method of § 200 to illustrate ex. 3.

5. If the segments of the diagonals of a quadrilateral are reciprocally proportional, what is true?

6. If the segments of the diagonals of a quadrilateral are proportional, what is true?

7. Construct a fourth proportional to three given lines, using Prop. I.

8. Construct a fourth proportional to three given lines, using Prop. VI.

9. Construct a third proportional to two given lines.

NOTE. To complete the proof of the next proposition the student should carefully examine Prop. I, pages 90, 91.

PROPOSITION VIII. THEOREM.

254. *In equal circles, or in the same circle, two central angles are proportional to their intercepted arcs.*

Let the central angles O_1, O_2 intercept the arcs AB, AC.

It is to be proved that $\angle O_1 : \angle O_2 :: \text{arc } AB : \text{arc } AC$.

CASE I. When the arcs are commensurable.

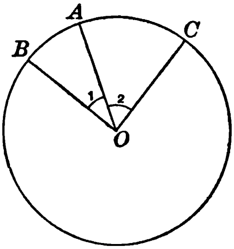


FIG. 107.

Let AB and AC be two commensurable arcs.

The common measure (m) will be contained in the arcs integral numbers of times.

$$\text{Arc } AB : \text{arc } AC = h : k.$$

Draw radii to the points of division.

$$\angle O_1 : \angle O_2 = h : k.$$

CASE II. When the arcs are incommensurable.

Let AB and AC be incommensurable arcs.

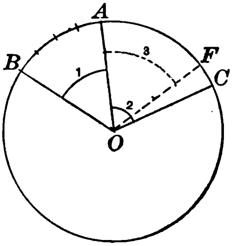


FIG. 108.

Divide AB into any (possible) number of equal parts.

$$\angle O_1 : \angle O_2 = \text{arc } AB : \text{arc } AF.$$

Divide AB into so many equal parts that one part shall be smaller than FC.

AF is made to approach AC as a limit.

$\angle O_1 : \angle O_3$ is made to approach $\angle O_1 : \angle O_2$ as a limit.

Arc AB : arc AF is made to approach arc AB : arc AC as a limit.

But the variable $\angle O_1 : \angle O_3$ always equals the variable arc AB : arc AF.

255. SCH. 1. The arc adopted as a unit for most practical purposes is one three-hundred-sixtieth of the circumference, and is called one degree (1°) of arc. The central angle intercepting this arc is called one degree of angle, and is a unit for angular measure. The proposition in § 254 establishes that any central angle is to 1° of angle as the intercepted arc is to 1° of arc. Therefore the *numbers* of degrees in any central angle and its intercepted arc are the same. And evidently if any other unit beside $\frac{1}{360}$ of the circumference were chosen, the *numbers* of units in arc and central angle would be the same. This fact is briefly expressed:

A central angle is measured by its arc.

The degree is divided into sixty minutes (60′), and a minute into sixty seconds (60″).

256. SCH. 2. As a central angle is measured by its intercepted arc, the theorems in §§ 172, 175–179 may be stated in a form of which the following is an example: *An inscribed angle is measured by one-half its intercepted arc.*

EXERCISES. 1. Prove (without referring to § 175) the angle between two chords intersecting within a circle is measured by one-half the sum of its intercepted arc and the arc intercepted by its vertical angle.

2. State and prove as in ex. 1 the §§ 176–179.

PROPOSITION A. PROBLEM.

257. *To divide a given line into segments proportional to given lines.*

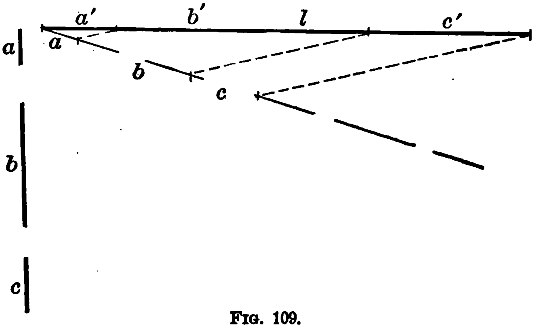


FIG. 109.

258. COR. *To divide a given line externally in a given ratio.*

EXERCISE. Divide a given line internally, and then externally, in the ratio of the side and diagonal of a square.

PROPOSITION B. PROBLEM.

259. *To construct the fourth proportional to three given lines.*

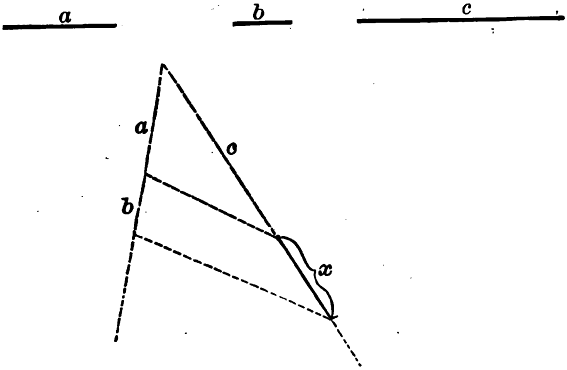


FIG. 110.

260. COR. *To construct the third proportional to two given lines.*

PROPOSITION C. PROBLEM.

261. *To construct the mean proportional between two given lines.*

EXERCISES. 1. How many earlier theorems involve a mean proportional? Find an *exercise* which will furnish a construction for § 261.

2. Divide a given line into three segments so that the second is a mean proportional between the other two.

3. Solve ex. 2 having the ratio of the first segment to the second a *given* ratio.

4. Find the mean proportional between the side and diagonal of a square.

262. DEF. A line is **divided harmonically** when it is divided externally and internally in the same ratio.

Thus by §§ 248, 248*a* the base of a triangle is divided harmonically; *i.e.*, internally and externally in the ratio of the sides.

$$PA : PB :: QA : QB.$$

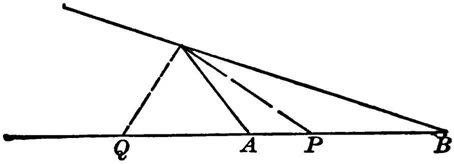


FIG. 111.

PROPOSITION D. PROBLEM.

263. *To divide a given line harmonically in a given ratio.*

[The figure will be clearer if the two constructions— (1) for *internal,* (2) for *external,* section—are made on opposite sides of the given line (§§ 257, 258).

The terms of a proportion involving sections of a line are conveniently read *from* the ends of the line *to* the points of section.]

264. COR. *If a line is divided harmonically, the extremities of the line divide harmonically the line between the points of section.*

If AB is divided harmonically at I and E, then IE is divided harmonically at A and B.

265. SCH. The points A, B; I, E of § 264 are *harmonic points,* and A, B are *harmonic conjugates* of I, E.

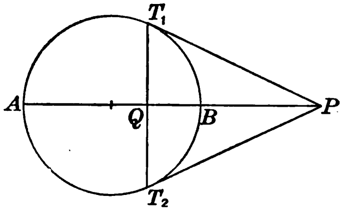


FIG. 112.

Ex. 1. Prove that A, B; Q, P are harmonic points.

2. Show how AB is harmonically divided at I, E.

3. Find E when I, A, B are given.

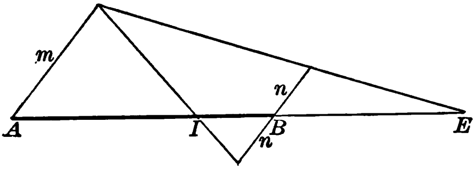


FIG. 113.

DEFS. If the points A, B (Fig. 112) are harmonic conjugates of P, Q, then A, Q, B, P form a *harmonic range.*

The *arithmetic mean* of two lines is one-half their sum.

The *geometric mean* of two lines is their mean proportional.

The *harmonic mean* of AQ, AP in the harmonic range A, Q, B, P is AB.

4. What is the arithmetic mean of PA, PB?
5. Of what is PT the geometric mean?
6. What is the harmonic mean of PA, PB?

266. Def. A line is divided in **extreme and mean ratio** when it is divided into two segments so that one segment is a mean proportional between the whole line and the other segment.

The line is divided **internally** or **externally** according as the point of section is on the line or on the line produced.

Proposition E. Problem.

267. *To divide a given line in extreme and mean ratio.*

Let AB be the line.

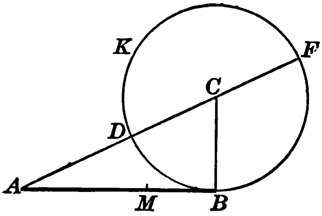


Fig. 114.

Bisect AB; $\left(\frac{AB}{2} = r\right)$.

At B draw BC perpendicular to AB equal to r.

With C as center draw the circle K.

Draw AC.

With A as center and AD as radius, lay off AM.

M is the point of section required.

$$AF : AB :: AB : AD.$$ [?

By *division,* § 222, IV, (2), $AD : AM :: MB : AD.$

By *inversion,* § 222, II, $AB : AM :: AM : BM.$

By using AF in place of AD in the construction the external point E may be found.

To prove the external section take

$$AB : AD :: AF : AB.$$

Use § 222, III, (1); then II.

Note. The division of a line in extreme and mean ratio is called The Golden Section.

Exercises. 1. A divides the line EB in golden section.

2. To find the whole line when the shorter section is given.

3. To find the whole line when the longer section is given.

PROPOSITION F. PROBLEM.

268. *To construct a triangle similar to a given triangle with any given ratio of similitude.*

Let ABC be the given triangle and let $l : n$ be the ratio of similitude.

Construct h a fourth proportional to l, n, a.

At one extremity of h construct $\angle J = \angle C$.

At the other extremity of h and on the same side as is $\angle J$, construct $\angle I = \angle B$.

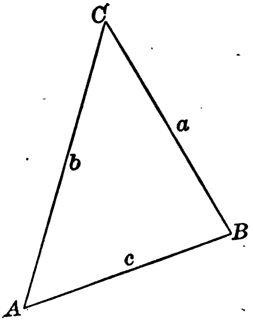


FIG. 115.

The sides of the angles drawn will intersect at H.

HIJ is the triangle required. [?

269. COR. *To construct a triangle similar to a given triangle on a given line as base.*

EXS. 1. To inscribe in a given circle a triangle similar to a given triangle.

2. To circumscribe about a given circle a triangle similar to a given triangle.

DEF. The point of concurrence of the medians of a triangle is called the *centroid* of the triangle.

3. *If perpendiculars are drawn from the centroid and the vertices of a triangle to any line outside the triangle, the perpendicular from the centroid is equal to one third the sum of the perpendiculars from the vertices.*

PROPOSITION G. PROBLEM.

270. *To construct a polygon similar to a given polygon with a given side as base.*

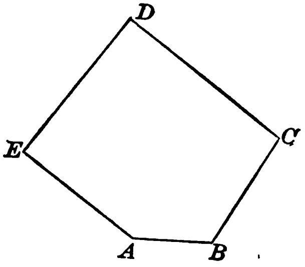


FIG. 116.

Let $ABCDE$ be the given polygon.

Let HI

be the side homologous to AB.

It is required to construct on HI a polygon similar to $ABCDE$ (P).

Draw the diagonals from A dividing P into triangles ABC, ACD, etc.

On HI as the homolog of AB construct triangle 1, similar to triangle ABC.

Adjacent to *1* construct triangle *2*, similar to triangle ACD; and so on.

Then will the resulting polygon (Q) be the polygon required.

For $\angle I = \angle B$, $\angle J = \angle C$, $\angle K = \angle D$, etc., and the homologous sides are in the ratio of similtude $AB : HI$.

271. THEOREM. *If two polygons are composed of the same number of triangles similar each to each and similarly placed, the polygons are similar.*

272. THEOREM. *If two polygons are similar, they may be resolved into the same number of triangles, etc.*

273. THEOREM. *The perimeters of two similar polygons are in the ratio of similitude.*

274. SCH. 1. In two similar polygons any two homologous lines are in the ratio of similitude. This can generally be proved in any particular case by similar triangles involving the two homologs.

275. SCH. 2. From the solution of Problem G (§ 270) it is evident that *two* polygons, Q and Q', may be constructed on HI, each similar to P, but differing in the direction in which the homologous angles occur. In Q the angles H, I, J, etc., occur counter-clockwise, as their equals do in P. In Q', the angles I, H, J, etc. (equal respectively to A, B, C, etc.), occur clockwise.

Q is said to be *directly* similar, and Q' *inversely* similar, to P.

NOTE. It should be noticed that the constructions are generally *unique, i.e.*, that with given parts only one construction is possible; for example (§ 259), there is only one fourth proportional to three given lines. Section 275 makes no exception to this comment, for the polygon Q' may be made to coincide with Q by turning Q' "face down," and superposing it upon Q.

EXERCISES.

1. The perpendicular from any point in a given arc to the chord of the arc is a mean proportional between the perpendiculars from the same point to the tangents at the extremities of the arc.

2. *The three common chords of three intersecting circles are concurrent.*

3. A right triangle is inscribed in a semicircle, a tangent is drawn at the vertex of the right angle, and perpendiculars are drawn from the other vertices to the tangent. Prove each side of the triangle a mean proportional between the diameter and a perpendicular.

4. In the trapezoid $ABCD$ the non-parallel sides DA, BC intersect at O, and the diagonals intersect at Q Prove OQ a median of the triangle OCD.

5. In a circle (center O) TA is a tangent at T, and TP is the perpendicular from T to AO. Prove $AO : OT :: OT : OP$.

6. A tangent and a secant are drawn from a point to a circle. Prove that the perpendicular from the point of tangency to the secant is a mean proportional between the perpendiculars from the points of intersection to the tangent.

7. Let C be the vertex of a triangle on a given base AB, having its sides a, b in a given constant ratio, $m : n$. Show that as a and b vary in length the bisectors (internal and external) of $\angle C$ inclose a

constant angle and pass through fixed points on AB. Find the locus of C. [§ 248.]

8. *K is a given circle and A, B are two given points, S is a variable circle always passing through A and B, and intersecting K.*

(1) *Prove that the common chords of S and K are concurrent;*

(2) *Find the locus of the center of the common chord of S and K.*

9. In a given right triangle to inscribe a square with one side on the hypotenuse. [See ex. 3, p. 102.]

10. In a given right triangle to inscribe a square with one side on the base.

11. In a given triangle to inscribe a square with one side on the base of the triangle. [See ex. 13, p. 96. Take ACD, Fig. 105, for the given triangle, and call BCD a right angle. Use ex. 10, p. 108.]

12. Through the point of intersection of two circles to draw a secant whose chords in the two circles are equal. [Bisect the line of centers and join that point to the point of intersection.]

13. To construct a circle passing through two given points and tangent to a given line. [§ 253.]

14. To construct a circle passing through a given point and tangent to two given lines.

15. To construct a circle tangent to a given circle and also tangent to two given lines.

REGULAR POLYGONS.

276. Def. A **regular polygon** is a polygon which is both equilateral and equiangular.

The square is the regular quadrilateral, and the equilateral triangle is the regular triangle.

Proposition H. Problem.

277. *To inscribe a square in a given circle.*

278. Sch. It will be easily seen that the problem of constructing a regular polygon of a given number of sides is equivalent to the problem of inscribing an equilateral polygon of the given number of sides in a circle.

This problem, "To inscribe in a circle an equilateral polygon," can be accomplished if the circumference can be divided into the required number of equal parts, or if four right angles (about the center) can be divided into the required number of equal angles.

Note. When two similar polygons are regular, the distinction in § 275 between Q and Q' disappears.

Exercises. 1. If the vertices of an equilateral polygon are concyclic, the polygon is regular.

2. To inscribe a regular octagon in a circle.

3. To construct a regular octagon on a given side.

PROPOSITION J. PROBLEM.

279. *To inscribe a regular hexagon in a given circle.*

Let K be the circle, radius r.

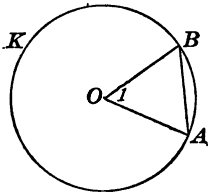


FIG. 117.

Lay off an arc a whose chord is the given radius.

Six such chords will form the hexagon required.

Draw the chord AB; join OA, OB.

In triangle OAB,

$$\begin{aligned}\angle O_1 &= \text{one-third of two right angles}\\ &= \text{one-sixth of four right angles.}\end{aligned}$$

$\therefore$ arc a = one-sixth of the circumference.

[Show that the polygon ABC, etc., is *regular.*]

280. COR. 1. *To inscribe a regular triangle in a given circle.*

281. COR. 2. *To inscribe a regular dodecagon in a given circle.*

EXERCISES. 1. Construct a regular dodecagon by joining twelve equal isosceles triangles.

2. How large is one vertical angle in a triangle of exercise 1?

3. How large is a vertical angle of one of the ten triangles forming a regular decagon?

4. In one of the isosceles triangles forming a regular decagon how large is a base angle? How does it compare with the vertical angle?

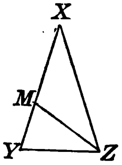


FIG. 118.

5. Suppose XYZ is an isosceles triangle in which $\angle Y = 2\ \angle X$, and suppose ZM bisects $\angle Z$.

(*a*) How many isosceles $\triangle$s in the figure?

(*b*) How many similar $\triangle$s?

(*c*) Write a proportion involving YM.

(*d*) How is XY divided at M?

6. In Fig. 118 how many degrees in $\angle XMZ$? What regular polygon would have $\angle XMZ$ for one of its angles?

PROPOSITION K. PROBLEM.

282. *To inscribe a regular decagon in a given circle.*

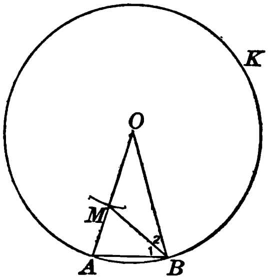


FIG. 119.

Let K be the circle whose radius is r.

Divide r in extreme and mean ratio, and let OM be the longer segment $(= s)$.

With A for center and s for radius, draw an arc intersecting the circumference at B.

Draw chord AB.

Ten such chords as AB will form the decagon required.

Draw OB, BM.

In triangles OAB and BAM,

$$OA : AB :: BA : AM, \qquad \text{[Const.}$$

and $\angle A$ is common.

$\therefore$ the triangles are similar. [?

[Why is triangle OMB isosceles? What follows? How large is $\angle O$?]

283. COR. 1. *To inscribe a regular pentagon in a given circle.*

284. COR. 2. *To inscribe a regular 20-sided polygon in a given circle.*

PROPOSITION M. PROBLEM.

285. *To inscribe a regular pentedecagon in a given circle.*

[How can the required central angle be constructed? Use two angles which have already been used.]

286. SCH. 1. From the preceding sections it follows that regular polygons may be constructed of the following numbers of sides:

$$\begin{array}{rrrlll}
3, & 6, & 12, & \ldots, & \text{or} & 3 \cdot 2^n, \\
4, & 8, & 16, & \ldots, & \text{or} & 2^n, \\
5, & 10, & 20, & \ldots, & \text{or} & 5 \cdot 2^n, \\
15, & 30, & 60, & \ldots, & \text{or} & 3 \cdot 5 \cdot 2^n,
\end{array}$$

where n is any positive integer.

"The construction of any new regular polygon not included in one of these series will give rise to a new series. Till the beginning of this century nothing was added to the knowledge of regular polygons as given by Euclid. Then Gauss, in his celebrated *Arithmetic,* proved that every regular polygon of 2^n+1 sides may be constructed if this number 2^n+1 be prime, and that no others can be constructed by elementary methods. This shows that regular polygons of 7, 9, 13 sides cannot thus be constructed, but that a regular polygon of 17 sides is possible." (Ency. Brit., Article "Geometry.")

287. SCH. 2. It is evident that the numbers in the above series are the number of equal parts into which a circumference may be divided.

NOTE. These possibilities and limitations of construction should be noticed:

(1) Without the use of parallels, a given line may be bisected, again bisected, and so on.

(2) Without the use of parallels, proportion, or measurement of angles, a given arc (or a circumference) or a given angle may be bisected, again bisected, and so on.

(3) By the use of parallels a given line may be divided into any number of equal parts (or into any number of segments proportional to given lines).

(4) By the use of parallels (§§ 101, 233), a circumference may be divided into equal parts *provided* the number of the parts falls in one of the series mentioned in § 286.

PROPOSITION N. PROBLEM.

288. *To circumscribe a regular polygon about a given circle.*

Let K be the given circle.

Divide the circumference into any possible number of equal parts.

At the points of section draw tangents to the circle.

EXS. 1. Construct the required polygon, so that its sides shall be tangent at the vertices of an inscribed polygon.

2. Construct the required polygon so that its sides shall be parallel to the sides of an inscribed polygon.

PROPOSITION P. PROBLEM.

289. *To circumscribe a circle about any regular polygon.*

Let $ABCDEF$ be a regular polygon. It is required to circumscribe a circle about the polygon.

Find the center (O) of the circle through A, B, C.

With O for center and OA for radius, draw the circle K.

K is the circumscribing circle required.

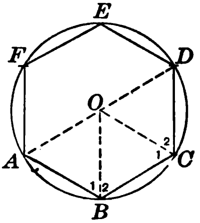


FIG. 120.

Draw OA, OB, OC, OD.

[Prove that the circle passes through D.]

PROPOSITION Q. PROBLEM.

290. *To inscribe a circle within any regular polygon.*

Circumscribe a circle about the polygon.

From the center, O, draw the perpendicular OP to a side AB.

Draw the circle with center O, radius OP.

This is the circle required.

EX. Prove this by drawing bisectors of two angles A, B, and showing that OC bisects angle C.

291. Cor. *The circumscribed and inscribed circles of a regular polygon have a common center.*

292. Defs. The **center** of a regular polygon is the center of its circumscribed and inscribed circles.

The **apothem** of a regular polygon is the perpendicular from center to side.

The **radius** of a regular polygon is the line from center to vertex.

The **central angle** of a regular polygon is the angle between two radii drawn to the extremities of a side.

Exercises. 1. A regular polygon of *k* sides (called a regular *k*-gon) may be resolved into *k* equal isosceles triangles.

2. The central angle of a regular polygon is the supplement of an angle of the polygon.

3. The central angle of a regular *k*-gon is one-*k*th of four right angles.

4. What is the formula for an angle of a regular *k*-gon?

Proposition IX. Theorem.

293. *Two regular polygons of the same number of sides are similar.*

294. Cor. 1. *The perimeters of similar regular polygons are proportional to their radii.*

294*a*. Cor. 2. *The perimeters of similar regular polygons are proportional to their apothems.*

295. Problem. *To circumscribe about a circle a polygon similar to a regular inscribed polygon and having its sides parallel to those of the given polygon.*

296. Cor. *The radii of a regular circumscribed polygon pass through the vertices of the similar inscribed polygon having parallel sides.*

SYMMETRY.

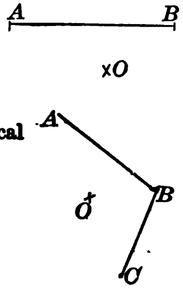
297. DEFS. A figure is **symmetrical with respect to a point** when that point bisects every line through it joining two points in the figure.

The **center of symmetry** is the bisecting point.

EXERCISES. 1. The intersection of the diagonals of a parallelogram is the center of symmetry of the parallelogram.

2. The center of a regular hexagon is the center of symmetry of the hexagon.

3. Construct the other part of a symmetrical figure of which O is the center.



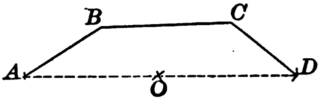


FIG. 121.

298. DEFS. A figure is **symmetrical with respect to a line** when that line bisects every line perpendicular to it joining two points in the figure.

The **axis of symmetry** is the bisecting line.

EXERCISES. 1. The median of an isosceles triangle is an axis of symmetry.

2. A diameter of a circle is an axis of symmetry.

3. How many axes of symmetry has a regular hexagon?

4. How many axes of symmetry has a regular pentagon? Has it a center of symmetry?

5. Does a rectangle have axial symmetry Does it have point symmetry?

6. When the required part of Fig. 121 has been constructed, does the complete figure have axial symmetry?

299. THEO. *If a figure is symmetrical with respect to two axes perpendicular to each other, it is also symmetrical with respect to their intersection as a center of symmetry.*

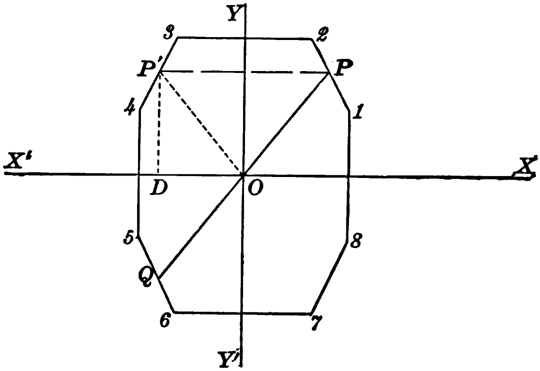


FIG. 122.

PQ is any line through O, and it is to be proved that $PO = QO$.

Draw $PP' \perp$ to YY', draw $P'D \perp$ from P' to $X'X$, intersecting *5–6* in Q'. Draw OQ'.

$OQ' = OP$, and OQ' falls on OQ. $\therefore OQ = OP$.

NOTE. Two figures are often called symmetrical (*i e.* with reference to each other) when they are the halves of a single geometrical figure. Thus, in Fig. 122, *1–2–3* and *5–6–7* are symmetrical with respect to O; also *2–1–8–7* and *3–4–5–6* are symmetrical with respect to YY'.

In what way may the two symmetrical figures be made to coincide?

SIMILAR FIGURES IN PERSPECTIVE.

300. 1. If two directly similar polygons have two homologous sides parallel, all the homologous sides are parallel.

2. If two similar polygons have their homologous sides parallel, the lines joining their homologous vertices are concurrent.

3. DEFS. The point of concurrence is called *the center of similitude* of the polygons. The polygons in this position are said to be *in perspective.*

4. If two similar polygons are in perspective, the center of similitude divides the line joining two homologous vertices in the ratio of similitude.

5. DEFS. If the point divides the line internally, the point is an *internal* center of similitude; if externally, an *external* center. When the center is internal the polygons are *oppositely* placed; when external they are *similarly* placed.

6. Let two similar polygons be in perspective; let any line S through the center of similitude O intersect the polygons in the homologous points P, P'. Then the segment PP' is divided in the ratio of similitude at O.

7. DEFS. For this reason two similar polygons which are in perspective are called *homothetic.*

8. If two similar polygons are in perspective, any two homologous lines in the polygons are parallel.

9. If two regular polygons of an even number of sides are in perspective they have two centers of similitude (E, I), external and internal. Show that I and E lie on the line joining the centers of the polygons. Suppose that the polygons have an odd number of sides.

10. If in any two circles (centers C, C') there are parallel diameters (PQ, $P'Q'$), the lines joining their extremities (PP', QQ') intersect the line joining the centers (CC') in a point (O) which divides the line of centers (CC') in the ratio of the radii of the circles.

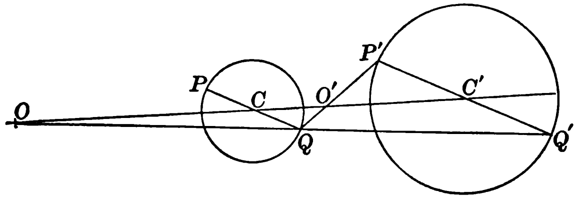


FIG. 123.

11. DEFS. O is the center of similitude of the circles. It is either external or internal as in 5 and 9 above; and the terms perspective, homothetic, etc., are applied to the circles as to similar polygons.

Consistently with these theorems all circles are regarded as ***similar figures.***

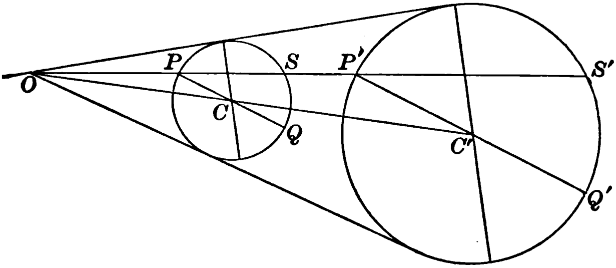


FIG. 124.

12. The tangents to two circles intersect in the center of similitude.

13. If a secant to two circles is drawn through their center of similitude (cutting the first in P, S,—the second in P', S'), the four segments from the center of similitude are in proportion. (The radii to the points of intersection are parallel in pairs.) (Fig. 124.)

14. DEFS. P, P' (extremities of parallel radii) are homologous points; so are S, S'. P, S' (extremities of non-parallel radii) are *anti-homologous*; so are P', S.

15. If two secants are drawn through the same center of similitude, two antihomologous points on one secant are concyclic with two antihomologous points on the other.

16. Show what the circle through $APS'B'$ becomes when the two secants OS', OB' coincide.

17. If the tangents from two antihomologous points (A, B') intersect (at H), the two tangents (HA, HB') are equal.

18. If a variable circle is always tangent to two fixed circles (at T, T'), the line joining the points of tangency (TT') passes through the center of similitude of the two fixed circles.

19. To draw a circle through two given points tangent to a given circle.

[Draw a circle through P, Q *intersecting* the given circle. Then use § 261 and § 253.]

20. To draw a circle through a given point (P) tangent to two given circles (C, C').

[Suppose the ⊙ drawn tangent at T, T' through P and intersecting OP at X. Then T, T', P and X are concyclic. But P, X, and any two antihomologous points on the given circles are concyclic. ∴ OX can be found. Then use 19 above.]

21. To draw a circle tangent to three given circles. [Call the given centers C_1, C_2, C_3. Let $r_1 > r_2 > r_3$. Draw ⊙ with center O_1, radius $= r_1 - r_3$ (S_1). Draw ⊙ with center O_2, radius $= r_2 - r_3$ (S_2). Draw ⊙ through O_3 tangent to S_1 and S_2. Then draw required circle.]

This problem is called The Problem of Apollonius.

NOTE. 1. The circle has several properties analogous to properties of the regular polygon. Let the number of sides of a regular inscribed polygon be doubled indefinitely, and let two consecutive sides—vertex A—be extended to secants. Then the angle A increases, the sides shorten, the vertices approach, and the secants come nearer to coincidence and to tangency. The circle is sometimes called a polygon with an infinite number of sides.

NOTE. 2. Before leaving the subject of proportional lines the student should carefully note the use of the symbols for lines.

(1) "The line a" sometimes means a fixed line of indefinite length, the symbol a simply marking its position. Thus "The altitudes of a triangle are concurrent" implies that the altitudes are prolonged as far as we please. If x denotes an altitude, x is a line fixed in position and indefinitely long.

(2) Sometimes "the line a" means an unmeasured length. Thus in "The altitude on the hypotenuse of a right triangle is a mean proportional between the segments of the hypotenuse," the altitude may be denoted by p, which is *not* indefinitely long, but begins at the vertex of the right angle and stops at the hypotenuse.

(3) Sometimes a, b, etc., denoting simply lengths are extended to mean the numerical measures of those same lengths. In this case the symbols are really the symbols of algebra. It is often convenient to pass from geometry to algebra, and back again to geometry; but it is well to keep carefully in mind just what the symbols mean and not to confuse different meanings.

BOOK III.

AREAS. MEASUREMENT.

AREAS.

301. DEF. **The area** of a figure is the amount of surface in it.

302. DEF. **Equivalent figures** are figures containing the same area.

303. DEFS. **The rectangle on two lines** is the rectangle having one line for the base and the other for the altitude of the rectangle.

The base and the altitude are called the **dimensions** of the rectangle.

A rectangle is also said to be *contained* by the two lines forming its base and altitude.

PROPOSITION I. THEOREM.

304. *Two parallelograms having equal bases and equal altitudes are equivalent.*

[Subtract equal triangles from the trapezoid.]

Note the case when one parallelogram is a rectangle.

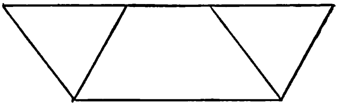


FIG. 125.

305. Cor. *Two triangles having equal bases and equal altitudes are equivalent.*

Exercises. 1. What is the locus of the vertex of a triangle of constant area on a fixed base?

2. A median of a triangle divides it into two equivalent triangles.

3. If two triangles abc, $a'b'c'$ have $a = a'$, $b = b'$, $\angle C$ the supplement of $\angle C'$, the two triangles are equivalent.

4. If two equivalent triangles have a common base and have their vertices on opposite sides of the base, the line joining their vertices is bisected by the base (produced if necessary).

5. In the parallelogram $ABCD$ any point, P, in the diagonal AC is joined to B and D. Prove the triangles ABP, ADP equivalent.

Proposition II. Theorem.

306. *A triangle is equivalent to one-half of a rectangle having for dimensions the base and altitude of the triangle.*

307. Cor. 1. *A trapezoid is equivalent to a rectangle having for dimensions the mid-parallel and the altitude of the trapezoid.*

308. Cor. 2. *A regular polygon is equivalent to one-half of a rectangle having for dimensions the perimeter and the apothem of the polygon.*

Exercises.

Def. The intersection of the diagonals of a parallelogram is called its center.

1. Any line through the center of a ▱ divides it into two equivalent figures.

2. A trapezoid is bisected by the line joining the middle points of its bases.

3. If through any point in the diagonal of a ▱ lines are drawn parallel to the sides, the parallelograms formed on opposite sides of the diagonal are equivalent.

4. In the trapezoid $ABCD$ a base is CD, and M is the mid-point of side BC. Prove that the triangle AMD is equivalent to one-half the trapezoid.

PROPOSITION A. PROBLEM.

309. *To construct a triangle equivalent to a given polygon.*

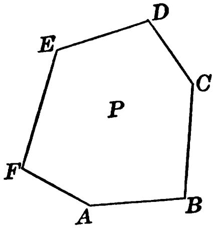


FIG. 126.

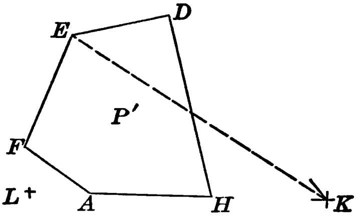


FIG. 127.

Draw BD. Through C draw CH parallel to DB intersecting AB in H. Draw DH.

Then $AHDEF$ is equivalent to P and has one less side than P has.

Continue similar constructions until a triangle (EKL) is obtained.

310. PROBLEM. *To construct a rectangle equivalent to a given polygon.*

311. THEOREM. *The square on the sum of two lines is equivalent to the sum of the squares on the lines plus twice the rectangle on the lines.*

312. COR. *The square on the difference of two lines is equivalent to the sum of the squares on the lines minus twice the rectangle on the lines.*

313. THEOREM. *The rectangle on the sum and the difference of two lines is equivalent to the difference of the squares on the lines.*

NOTE. The symbol ≎ is used for "equivalent."

PROPOSITION III. THEOREM.

314. *The square on the hypotenuse of a right triangle is equivalent to the sum of the squares on the other two sides.*

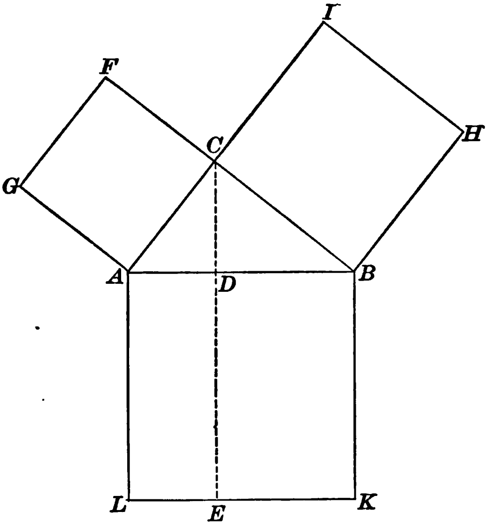


FIG. 128.

[Prove FCB a straight line. $\square AF \Bumpeq 2 \triangle GAB = 2 \triangle CAL \Bumpeq \square LD$. Also, $\square BI \Bumpeq \square BE$.]

314*a*. *The square on one side of a right triangle is equivalent to the difference of the squares on the hypotenuse and the other side.*

EXERCISES. 1. *Construct a square equivalent to the sum of two given squares.*

2. *Construct a square equivalent to the sum of three given squares.*

3. *Construct a square equivalent to the difference of two given squares.*

4. Show a relation among the second powers of certain numbers which is analogous to Prop. III.

Proposition IV. Theorem.

315. *The square on the mean proportional between two lines is equivalent to the rectangle on the lines.*

[Use §§ 314*a*, 313.]

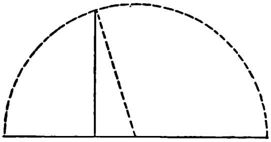


Fig. 129.

Proposition B. Problem.

316. *To construct a square equivalent to a given rectangle.*

317. Cor. 1. *To construct a square equivalent to a given triangle.*

318. Cor. 2. *To construct a square equivalent to a given polygon.*

319. Def. **The projection of a point upon a line** is the foot of the perpendicular from the point to the line.

320. Def. **The projection of one line upon another** is the length between the projections of the extremities of the first line upon the second.

Proposition V. Theorem.

321. *In any triangle the square on the base opposite an acute angle is equivalent to the sum of the squares on the two sides minus twice the rectangle on one side and the projection of the other side upon it.*

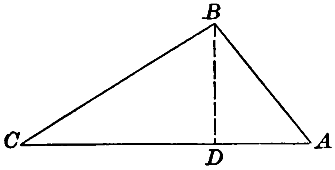


Fig. 130.

[Take AB for base. Call $AD = AC - CD$. Use §§ 314, 312.]

322. COR. 1. *In any obtuse-angled triangle the square on the base opposite the obtuse angle is equivalent to the sum of the squares on the two sides plus twice the rectangle on one side and the projection of the other side upon it.*

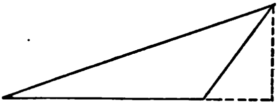


FIG. 131.

323. COR. 2. *If the sum of the squares on two sides of a triangle is equivalent to the square on the third side, the triangle is right-angled.*

EXS. 1. State §§ 314, 321. 322 in one general theorem.

2. State converses of §§ 321, 322.

3. State the "character" of a triangle (*i.e.*, whether acute-, right-, or obtuse-angled) when the sides are:

(1) 5, 7, 9;
(2) 6, 8, 10;
(3) 7, 9, 11.

4. *The square on the sum of two lines plus the square on the difference of the lines is equivalent to twice the sum of the squares on the lines.*

5. *The square on the sum of two lines minus the square on the difference of the lines is equivalent to four times the rectangle on the lines.*

324. THEO. *The sum of the squares on two sides of a triangle is equivalent to twice the square on half the base plus twice the square on the median.*

325. THEO. *The sum of the squares on the four sides of a parallelogram is equivalent to the sum of the squares on the diagonals.*

EXERCISE. *The sum of the squares on the four sides of a quadrilateral is equivalent to the sum of the squares on the diagonals plus four times the square on the line joining the middle points of the diagonals.*

326. THEO. *The difference of the squares on two sides of a triangle is equivalent to twice the rectangle on the base and the projection of the median upon the base.*

PROPOSITION VI. THEOREM.

327. *The rectangle on the extremes of four proportional lines is equivalent to the rectangle on the means.*

[See ex. 2, page 98. Draw the four lines as segments of intersecting chords. Through the point of intersection draw the chord ⊥ to the diameter through that point. Use §§ 250, 315.]

328. COR. *To construct a rectangle equivalent to a given rectangle on a given line as base.*

329. SCH. *If through any given point within (or outside) a circle any number of chords be drawn, the area of the rectangle on the segments of a chord is constant.*

330. THEO. *The rectangle on two sides of a triangle is equivalent to the rectangle on the diameter of the circumscribed circle and the altitude of the triangle.* [Use § 327.]

331. THEO. *The rectangle on two sides of a triangle is equivalent to the rectangle on the segments of the base made by the bisector of the vertical angle plus the square on the bisector.*

[Use proportion from the triangles *BCF*, *BDA*. Also § 329.]

EXERCISE. *State and prove the analogous theorem for the bisector of the exterior vertical angle.*

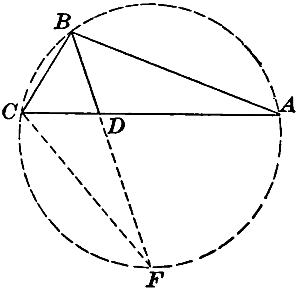


FIG. 132.

332. PROBLEM. *To construct a rectangle equivalent to a given square when the sum of the dimensions of the rectangle is given.*

[Use § 315.]

Ex. Show how to construct a rectangle equivalent to a given polygon when the sum of the dimensions of the rectangle is given.

333. PROBLEM. *To construct a rectangle equivalent to a given square when the difference of the dimensions of the rectangle is given.*

[Use § 253.]

NOTE.—"The area of a figure" is sometimes referred to simply as "the figure" when no ambiguity can occur.

PROPOSITION VII. THEOREM.

334. *Two rectangles with equal altitudes are proportional to their bases.*

[When AB and BC are commensurable draw perpendiculars to AC at the points of division and prove

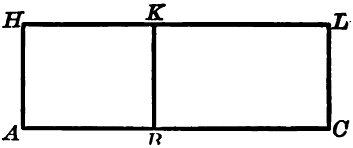


FIG. 138.

rectangle AK : rectangle BL :: AB : BC. See page 90.

When AB and BC are incommensurable divide AB into any number of equal parts, and use the equal variable ratios which approach limits. See page 91.]

335. Cor. 1. *Two rectangles with equal bases are proportional to their altitudes.*

336. Cor. 2. *Two triangles with equal altitudes are proportional to their bases.*

337. Cor. 3. *Two triangles with equal bases are proportional to their altitudes.*

Exercise. Two similar rectangles are proportional to the squares on homologous sides.

338. Theo. *If four lines are in proportion, the squares on the lines are in proportion.*

[Construct two rectangles having the lines as dimensions.]

339. Theo. *Two similar triangles are proportional to the squares on their homologous sides.*

Proposition VIII. Theorem.

340. *Two similar polygons are proportional to the squares on their homologous sides.*

[Divide the similar polygons P, P' into triangles as in § 272. Calling the triangles a, b, etc., show that $a : a' = \square\, s : \square\, s' = b : b'$, etc. Use § 222, VI.]

340*a*. Sch. Two similar polygons can in general be proved proportional to the squares on any homologous lines.

341. Cor. 1. *Two similar regular polygons are proportional to the squares on their radii.*

342. Cor. 2. *Two similar regular polygons are proportional to the squares on their apothems.*

343. Cor. 3. *The polygon whose base is the hypotenuse of a right triangle is equivalent to the sum of the polygons similar to it whose bases are the two sides of the triangle.* [Use §§ 340, 222, VI., and 314.]

PROPOSITION IX. THEOREM.

344. *The squares on the sides of a right triangle are proportional to the projections of those sides upon the hypotenuse.*

[Use Fig. 128.]

345. COR. *The similar polygons whose bases are the sides of a right triangle are proportional to the projections of those sides upon the hypotenuse.*

346. PROBLEM. *To construct two lines in the ratio of the areas of two given polygons.*

[Use §§ 318 and 344.]

Ex. 1. To construct a fourth proportional to two given polygons and a given line.

2. To construct a fourth proportional to two given lines and a given square (or a given polygon).

3. To construct as the fourth proportional in ex. 2 a polygon similar to the given polygon.

PROPOSITION C. PROBLEM.

347. *To construct a polygon similar to one given polygon and equivalent to another given polygon.*

Let P be the given polygon similar to the required polygon X. Let Q be the given polygon whose area equals that of the required polygon X.

[Use § 346 to get the ratio of areas of P and X $(= P : Q)$. Use ex. 3 above to get a polygon similar to P and also the required fourth proportional.]

EXERCISE. 1. To construct a third proportional to two given similar polygons which shall be similar to the given polygons.

2. The similar triangles ABC, $AB'C'$ have A common and BC parallel to $B'C'$. Prove $\triangle ABC'$ a mean proportional between ABC and $AB'C'$.

3. To construct a mean proportional between two given similar triangles which shall be similar to the given triangles.

4. To construct a rectangle equivalent to a given square when the ratio of the dimensions of the rectangle is given.

MEASUREMENT OF SURFACES.

348. The unit of surface measure is usually a square whose side is a linear unit.

Consider first the measurement of a rectangle when a given length l is the linear unit.

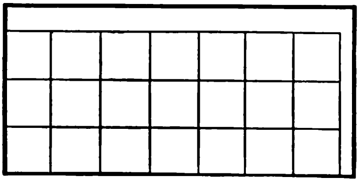


FIG. 134.

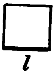


FIG. 135.

If we apply the square to the rectangle as many times as possible, there will in general be a margin of the rectangle on two sides not covered by the square units. Thus, this rectangle contains the square unit 21 times with a margin, and the measure of the rectangle is more than 21 and less than 32 square units.

Now by dividing the linear unit into equal parts a smaller square, which will be a known fraction of the square unit, may be applied to the margin, and in this way we may *approximate* the true measure of the rectangle. It should be observed (1) that what was said about measurement in section 202, and was there illustrated by lengths, applies equally to squares when a square is the unit; and (2) that as any polygon may be replaced by an equivalent square (§ 318), the remarks in section 202 may be extended to the case of any rectilinear surface.

PROPOSITION X. THEOREM.

349. *The number of square units in any rectangle is equal to the product of the numbers of linear units in its dimensions.*

Let R be any rectangle, and let S be the square unit whose side (l) is chosen as the linear unit.

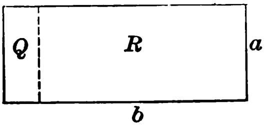


FIG. 136.

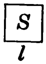


FIG. 137.

Let a, the altitude of R, contain l k times; and let b, the base of R, contain l n times.

It is to be proved that R contains S $k . n$ times.

Construct a rectangle Q with base l and altitude a.

Then $Q : S = a : l$; [?

or Q contains S as many times as a contains l.

$$\therefore \; Q = k . S.$$

Also

$$R : Q = b : l;$$
$$\therefore \; R = n . Q.$$
$$\therefore \; R = n . k . S.$$
$$\therefore \; R \text{ contains } S \; k . n \text{ times.}$$

350. SCH. If a and b denote two lengths (see page 119, note 2), then "the product of a and b" has no meaning, but "the rectangle on a, b" has the meaning given in the definition § 303. But if a and b are assumed to denote two numbers, then "the product of a and b" has a meaning in algebra, while "the rectangle on a, b" has no meaning.

If a and b are used to denote sometimes the lengths of two lines, and sometimes the measures of those lines when referred to the same unit, we may state § 349:

The product of a and b (meaning measures of the lines) *is equal to the measure of the rectangle on a, b* (meaning lengths).

Or, briefly:

The area of a rectangle is equal to the product of its base and altitude.

$$(\square = a \,.\, b.)$$

In the same way § 306 gives:

The area of a triangle is equal to one-half the product of its base and altitude.

$$\left(\triangle = \frac{a \,.\, b}{2}.\right)$$

§ 307 is, when restated: *The area of a trapezoid is equal to the product of its mid-parallel and altitude.*

$$\left(\text{trapezoid} = a \,.\, \frac{b + b'}{2}.\right)$$

This use of the phrases "the product of two lines" and "the square of a line" is very common in geometry.

The student should restate §§ 308, 315, 321, 322, 326, 327, 329, 330, 331.

EXERCISES.

1. What is the area of a rectangle whose base is $\sqrt{3}$ and whose diagonal is $\sqrt{7}$?

2. What is the diagonal of a rectangle whose dimensions are 10 and a?

3. What is the area of a rectangle whose perimeter is 100 and whose length is four times its width?

4. What is the greatest rectangular area that a string one yard long can enclose?

5. A circle whose diameter is 1 meter has chords drawn through a point P. If the area of the rectangle on the segments of the diameter equals 16 square decimeters, how far is P from the center?

6. If the diameter is 6 inches and the area (as in ex. 5) is 16 sq. in., how far is P from the center?

7. If a regular hexagon has an area of 1 foot, what is the length of one side?

8. What is the area enclosed by a string one foot long—
   (1) when in the form of a regular triangle?
   (2) " " " " " " " hexagon?
   (3) " " " " " " " octagon?

9. To construct the base of a triangle, given the area and the altitude.

10. To construct a triangle, given the area, the base, and one angle.

11. *If two triangles have an angle in one equal to an angle in the other, their areas are proportional to the products of the sides including the equal angles.*

[Place the △s with an angle common.]

12. Extend ex. 11 to two triangles having an angle in one supplementary to an angle in the other.

13. What is the area of a regular decagon whose radius is 1?

14. What is the area of a regular pentagon whose radius is 1?

15. In the regular pentagon *ABCDE* whose radius is 1, what is the area of the trapezoid *BCDE*?

16. What is the apothem (1) of a square,
   (2) of a regular pentagon,
   (3) " " " hexagon,

the perimeter in each case being 10 inches? How do the areas of the three figures compare?

EXERCISES IN MAXIMA AND MINIMA.

NOTE. It is assumed of the figures in elementary geometry that in a given class of figures there is one and only one maximum, and there is one and only one minimum.

1. Of the parallelograms with two given sides, that which is a rectangle is the maximum.

2. *Of the triangles with two given sides, that in which the sides are perpendicular is the maximum.*

3. *Of the triangles on a given base and with a given perimeter, that which is isosceles is the maximum.*

4. Of the triangles on a given base and with a given area, that which is isosceles has the min. perimeter.

Def. Isoperimetric figures are figures having equal perimeters.

5. Of isoperimetric triangles, that which is equilateral is the max. [Use indirect method.]

6. State and prove a converse of ex. 4.

7. *Of the isoperimetric polygons having the same number of sides, the one which is the maximum is the equilateral polygon.* [Use ex. 3.]

8. *If all the sides of a polygon are given except one, the max. of such polygons is inscriptible in a semicircle having the undetermined side for a diameter.*

[Show that if some vertex (D) were not on the semicircumference, the polygon would not be max. Use ex. 2.]

9. *If all the sides of a polygon are given, that polygon which is inscriptible is the max.*

[Let $ABCD$ . . . (or P) be inscriptible. Let $A'B'C'$ . . . (or P') be mutually equilateral to P, but not inscriptible.

Draw diameter AM (cutting side DE) and construct $\triangle$ DME. Also construct the equal $\triangle$ $D'M'E'$.

Then polygon $MB > M'B'$, while polygon $MF =$ polygon $M'F'$,
or $MB = M'B'$, while $MF > M'F'$,
or else $MB > M'B'$, while $MF > M'F'$.]

10. *Of the isoperimetric polygons having the same number of sides, the one which is the maximum is the regular polygon.* [Use exs. 9, 7.]

11. If a polygon is regular, it is the max. polygon having that perimeter and that number of sides.

12. *Of two regular isoperimetric polygons, that one which has the greater number of sides is the greater.*

13. (Use Note 1, p. 119.) *Of the figures having a given perimeter, the circle is the maximum.*

14. *Of two regular polygons having a given area, that one having the greater number of sides has the less perimeter.*

15. *The circumference of a circle is less than the perimeter of any polygon having the same area.*

MEASUREMENT OF THE CIRCLE.

351. Axiom V. *The circumference of a circle is less than the perimeter of any polygon circumscribed about it.*

Proposition XI. Theorem.

352. *If the number of sides of a regular inscribed polygon is indefinitely increased, the apothem approaches the radius as its limit.*

Let P denote the regular polygon inscribed in a circle, let $a =$ apothem, $r =$ radius, and $x =$ one side of the polygon.

Let the number of sides be indefinitely increased by doubling the number.

$$a \text{ is greater than } r - \frac{x}{2},$$

and a is less than r.

When the number of sides is indefinitely increased, one of the equal arcs subtended by x approaches zero as a limit.

$\therefore \frac{x}{2}$ approaches zero as a limit.

$\therefore a$ approaches r as a limit.

Exs. 1. If the number of sides of a regular circumscribed polygon is indefinitely increased, the radius approaches the apothem as limit. [Use method of Prop. XI.]

2. If the number of sides of a regular circumscribed and of a similar inscribed polygon is indefinitely increased, the ratio of the perimeters approaches unity.

PROPOSITION XII. THEOREM.

353. *If the number of sides of a regular circumscribed polygon is indefinitely increased, the perimeter approaches the circumference as its limit.*

Let p denote the perimeter of the regular polygon circumscribed about the circle, and let q denote the perimeter of a polygon inscribed in the circle and always similar to the circumscribed polygon; also let r and a denote their respective apothems. Let c denote the circumference of the circle.

It is to be proved, etc.

$$p : q = r : a;$$

$$\therefore p - q : p = r - a : r;$$

$$\therefore p - q = \frac{r - a}{r} \cdot p.$$

Now let the number of sides of the polygons be indefinitely increased. In the right-hand side of the last equation the numerator of the fraction approaches zero, [?
the denominator (r) is constant,
and p decreases;

$\therefore$ the right-hand side approaches zero as a limit.

$\therefore$ the difference between p and q approaches zero as a limit.

But c is greater than q and less than p; [?

$\therefore$ the difference between p and c approaches zero as a limit.

But c is a constant;

$\therefore$ p approaches c as a limit.

Ex. Prove Prop. XII by using the two exercises preceding it.

354. COR. 1. *If the number of sides of a regular inscribed polygon is indefinitely increased, the perimeter approaches the circumference as its limit.*

355. Cor. 2. *The circumferences of circles are proportional to their radii.*

$$p : p' = r : r';$$

$$\therefore\ p = \frac{r}{r'} \cdot p';$$

$$\therefore\ c = \frac{r}{r'} \cdot c'; \quad \text{or} \quad \frac{c}{c'} = \frac{r}{r'}.$$

355*a*. Cor. 3. *The circumferences of circles are proportional to their diameters.*

356. Sch. The ratio of circumference to diameter is constant.

357. Def. The **ratio of circumference to diameter** is denoted by π.

(Circumference $= \pi D = 2\pi r$.)

Proposition XIII. Theorem.

358. *If the number of sides of a regular circumscribed polygon is indefinitely increased, the area approaches the area of the circle as its limit.*

Let P be a regular polygon circumscribed about the circle C, and let Q be a similar inscribed polygon whose sides are parallel to those of P.

It is to be proved, etc.

$(P - Q)$ is equal to the sum of the trapezoids enclosed between the two perimeters.

$\therefore\ (P - Q) < \square\, p \,.\, (r - a)$.

$\therefore\ P - Q$ approaches zero when the number of sides is indefinitely increased.

$\therefore\ P$ approaches C as a limit.

359. Cor. 1. *If the number of sides of a regular inscribed polygon is indefinitely increased, the area approaches the area of the circle as its limit.*

360. Cor. 2. *The areas of circles are proportional to the squares on their radii.*

360*a*. Cor. 3. *The areas of circles are proportional to the squares on their diameters.*

361. Sch. All circles are similar figures.

This scholium is consistent with the description of similar figures as those having the same shape, and also with the fact that circles are limiting cases of similar polygons.

361*a*. Defs. **Similar sectors** are sectors having equal central angles.

Similar segments and **similar arcs** are referred to equal central angles.

362. Cor. 4. *Similar sectors are proportional to the squares on their radii.*

363. Sch. The circle is a regular figure.

This scholium is consistent with the fact that circles are limits of regular polygons.

Exercises. 1. How many axes of symmetry has a circle?

2. Prove that the circle has a center of symmetry.

Proposition XIV. Theorem.

364. *The area of a circle is equal to one-half the product of its circumference and radius.*

[Use the area of a regular circumscribed polygon.]

$$\left(\odot = \frac{c \cdot r}{2} = \pi r^2.\right)$$

365. Cor. *The area of a sector is equal to one-half the product of its arc and radius.*

$$\left(\text{Sector} = \frac{s \cdot r}{2}.\right)$$

366. Problem. *To compute an approximate numerical value of the length of a chord when the length of the radius and the length of the chord of twice the arc are given.*

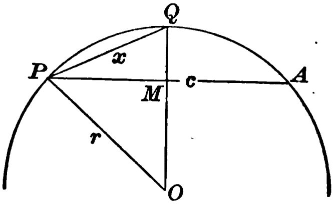


Fig. 138.

Let r denote the numerical length of the radius, and let c denote the numerical length of the given chord subtending the arc PQA.

It is required to compute the length of x subtending one-half the arc PQA.

$$x = \sqrt{\left(\frac{c}{2}\right)^2 + (r - \overline{OM})^2} = \sqrt{\left(\frac{c}{2}\right)^2 + \left(r - \sqrt{r^2 - \left(\frac{c}{2}\right)^2}\right)^2}$$

$$= r\sqrt{2 - \sqrt{4 - \frac{c^2}{r^2}}}.$$

Note. When $r = 1$, $x' = \sqrt{2 - \sqrt{4 - c^2}}$.

Exs. 1. When $c = 2$, $x' = ?$

2. When $x' = 1$, $c = ?$

3. In a circle whose radius is 1, compute the perimeters of the regular inscribed and circumscribed polygons of four, of eight, of sixteen sides.

4. As in ex. 3, compute the perimeters of polygons of six and of twelve sides.

5. Arrange in columns and in progressive order the results of exs. 3 and 4 and observe the approximate values of the circumference.

367. Problem. *To compute an approximate numerical value of* π.

$\pi = 3.14159 +$.

Note. For work in common fractions, $3\frac{1}{7}$ is a value for π often used. For work in decimals, 3.1416 is commonly used. π denotes the *precise* value of the ratio of circumference to diameter.

For a historical sketch of the problem of "The Squaring of the Circle" see an article by Hermann Schubert in "The Monist," Vol. I, No. 2.

368. Defs. A **circumangle** (or a **perigon**) is the whole angular magnitude about a point.

A **radian** is the central angle intercepting an arc equal in length to the radius.

Evidently π radians = 2 right angles; and the formula connecting the angular units is:

1 circumangle = 4 right angles = 360 degrees = 2π radians.

Although the radian cannot be constructed, it is a convenient unit in some branches of mathematics.

The radian is approximately equal to $57\frac{3}{11}°$.

Exercises. 1. Express in radians 60°, $\frac{1}{2}\,\underline{|R}$, 135°, 30°, 225°, 180°.

2. Express in degrees π radians, $\frac{\pi}{3}$ radians, $\frac{\pi}{4}$ radians, $\frac{3\pi}{2}$ radians.

3. A wheel whose radius is 21 inches rolls on a straight track until an arc of $\frac{7\pi}{4}$ radians has touched the track. How far has the wheel moved?

SUPPLEMENTARY PROPOSITIONS.

369. 1. Let p and q denote the lengths of the perimeters of a regular circumscribed and a similar inscribed polygon of n sides, and let p_2 and q_2 denote the lengths of perimeters of regular circumscribed and inscribed polygons of $2n$ sides. To compute (1) p_2 and (2) q_2 in terms of p and q.

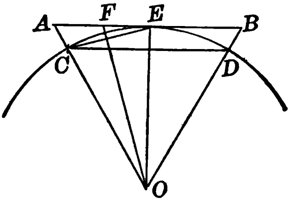


FIG. 139.

$$\frac{p}{q}=\frac{AF}{FE}; \quad \therefore \frac{p+q}{q}=\frac{AE}{FE}=\frac{2p}{p_2}.$$

$$\therefore p_2=\frac{2pq}{p+q}. \qquad (1)$$

$$\overline{CE}^2=\overline{CD}\,.\,\overline{CF}; \quad \therefore q_2=\sqrt{p_2\cdot q}. \qquad (2)$$

2. To compute the altitude of a triangle when the lengths of the three sides are given.

From § 321,

$$b^2=a^2+c^2-2aa_1;$$

$$\therefore a_1=\frac{a^2-b^2+c^2}{2a}.$$

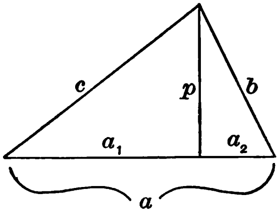


FIG. 140.

From § 314*a*,

$$p=\sqrt{c^2-a_1^2}=\sqrt{(c+a_1)(c-a_1)},$$

$$=\frac{\sqrt{(2ac+a^2-b^2+c^2)(2ac-a^2+b^2-c^2)}}{2a},$$

$$=\frac{\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}}{2a}.$$

If we denote $a+b+c$ by $2s$, this becomes

$$p=\frac{\sqrt{2s(2s-2a)(2s-2b)(2s-2c)}}{2a}$$

$$=\frac{2\sqrt{s(s-a)(s-b)(s-c)}}{a}.$$

Observe that the formula assumes the same form when § 322 is used.

3. To compute the area of a triangle when the lengths of the three sides are given.

Denote the area by K. Then

$$K=\frac{a}{2}\cdot p=\sqrt{s(s-a)(s-b)(s-c)}.$$

Observe what the formula becomes—

(1) When the triangle is isosceles ;

(2) When it is equilateral.

4. Show that the radius of the incircle is equal to $\frac{K}{s}$.

$$\left(r=\frac{K}{s}.\right)$$

5. If r_1 denotes the radius of the circle escribed to the side a, then

$$r_1=\frac{K}{s-a}.$$

6. If R denotes the radius of the circumcircle, then

$$R=\frac{abc}{4K}.$$

$[bc=p\,.\,2R, \quad \therefore\ abc=2ap\,.\,R=4K\,.\,R.]$

7. To find a formula for the length of a median of a triangle when the lengths of the sides are given.

[Use § 324.]

8. To find a formula for the segments of the base of a triangle made by the bisector of the vertical angle when the three sides are given.

[Use § 248.]

9. To find a formula for the length of the bisector of the vertical angle of a triangle when the three sides are given.

Denote the bisector of angle C by k. Use §§ 248 and 331 to obtain

$$ab = \frac{abc}{(a+b)^2} + k^2.$$

Then

$$k = \sqrt{ab\left(1 - \frac{c^2}{(a+b)^2}\right)}$$

$$= \frac{1}{a+b}\sqrt{ab(a+b+c)(a+b-c)}.$$

Using $2s = a + b + c$, we get

$$k = \frac{2}{a+b}\sqrt{abs(s-c)}.$$

10. In an inscribed quadrilateral, the product of the diagonals is equal to the sum of the products of the opposite sides. (See Fig. 141.)

Construct $\angle ECB = \angle DCA$. Draw EB.

$\triangle$s ADC and FBC are similar; $\therefore\ a : BF = x : c$, or $ac = x \cdot BF$.

$\triangle$s DFC and ABC are similar; $\therefore\ d : x = DF : b$, or $bd = x \cdot DF$.

$\therefore\ ac + bd = x(BF + DF) = xy$.

11. In the same quadrilateral $\dfrac{x}{y} = \dfrac{ab + cd}{ad + bc}$.

$ab + cd = 2R \cdot h$, where h denotes the sum of the altitudes of the

$\triangle$s DAB, DCB on the base DB; and $ad + bc = 2R \cdot k$ in $\triangle$s ABC, ADC.

$$\therefore \frac{ab + cd}{ad + bc} = \frac{h}{k} = \frac{x}{y}.$$

These formulas for xy and $\frac{x}{y}$ enable us to construct the inscribed quadrilateral when the four sides are given.

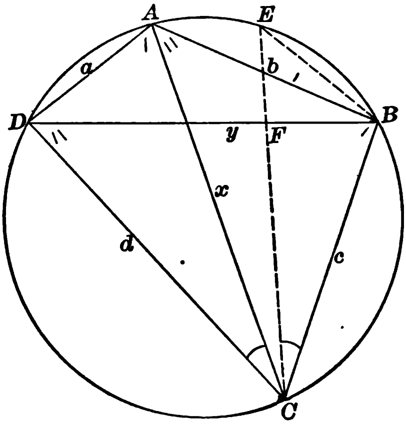


FIG. 141.

EXERCISES. Apply the preceding formulas to the following triangles:

1. $a = 6$, $b = 8$, $c = 10$.

2. $a = 5$, $b = 5\sqrt{3}$, $c = 10$.

3. $a = b = 8$, $c = 10$.

4. $a = 17$, $b = 22$, $c = 31$.

5. $a = 84.3$, $b = 127.5$, $c = 159.1$.

6. The sides of a quadrilateral are $a = 10$, $b = 16$, $c = 21$, $d = 13$. If it is to be inscribed in a circle, what are the lengths of its diagonals, and what is the radius of the circumscribed circle?

EXAMPLES.

1. If the eye of an observer is 21 ft. above the level of the sea, how far distant is his horizon, calling the radius of the earth 4000 miles?

2. If a clock is started at 12 o'clock, at what time has the tip of the minute-hand moved the length of the hand?

3. The diameter of a barrel-head is 21 inches, and the head consists of three pieces. The middle piece has for one straight edge the side of an inscribed square, and for the other straight edge the side of a regular inscribed triangle. What is the area of each piece?

4. A quarter mile running-track is to have straight sides and semicircular ends, and the straight part is to equal the curved part. What is the length of the plot?

5. The radius of a circle is 7 inches. Through a point 6 inches from the center a chord is drawn perpendicular to the diameter. Find the area of the smaller segment within $\frac{1}{10}$ of a square inch.

6. What is the radius of a circle in which the number of square units in its area equals the number of linear units in its circumference?

7. From a square card (side 1 in.) is to be cut a circle the area of which is one-half that of the square, and from a circular card (diameter 1 in.) is to be cut a square the area of which is one-half that of the circle. Find the radius of the first circle and the side of the second square.

8. A sector is to be rolled up into the shape of a cone. The base of the cone is 6 in. in diameter and the height is 4 in. Find (1) the radius of the sector, (2) its central angle, (3) its area.

9. A lawn is in shape a combined semicircle and trapezoid, the bases of the trapezoid being tangents to the semicircle at the extremities of the diameter. The front of the lawn is 85 ft. and the bases are 50 ft., 63 ft. Find the entire area and the perimeter.

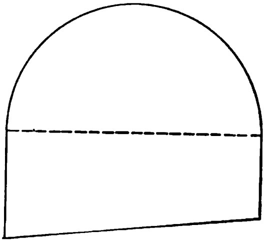


FIG. 142.

10. Seven equal circles are arranged, (1) tangent with their centers in a straight line, (2) six tangent circles grouped about a central circle. Which has the greater area—(1) a rectangle enclosing the first group, or (2) a circle enclosing the second group?

11. One angle of a triangle is $\frac{\pi}{3}$, and the including sides are a and b. Prove that the area is $\frac{ab}{4}\sqrt{3}$, and that the third side is $\sqrt{a^2 - ab + b^2}$.

12. Obtain a similar formula when the included angle is $\frac{\pi}{4}$.

13. What is the area of the largest semicircle that can be cut from a right triangle whose hypotenuse is 10 in. and one side 5 in.?

MISCELLANEOUS EXERCISES FROM COLLEGE ENTRANCE PAPERS.

1. In a triangle ABC, let O be the point in which the medians (lines drawn from the vertices to the middle points of the opposite sides) intersect. Prove that the triangles OAB, OAC, OBC are equivalent. (Amherst, 1894.)

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2. Given $\frac{x}{y} = \frac{a}{b}$, prove the following to be true or false, as the case may be. $\frac{x-y}{y} = \frac{a-b}{a}$; $x^2 : y^2 :: a^2 : b^2$. (Boston, 1894.)

3. A circular grass-plot is 40 feet in diameter; how many square yards in a concrete walk 4 feet wide surrounding the plot?

4. Construct a circle which shall be tangent to each of two parallel lines and whose circumference shall pass through a fixed point lying between the lines. Prove correctness of the construction.

5. Given an equilateral triangle, prove the radius of the inscribed circle equal to one-third the altitude.

---

6. The length of a tangent to a circle, from a point eight units distant from the nearest point on the circumference, is twelve units. Find the diameter of the circle. (Bowdoin, 1895.)

7. Find the ratio of the radius of a circle to the side of the inscribed square.

8. The area of a sector of sixty degrees is two hundred nine, and forty-four hundredths square inches. Find the length of the radius.

---

9. Define similar figures. Show that if two pairs of corresponding lines in two similar figures are parallel, then all pairs of corresponding lines are parallel.

A straight line is drawn cutting off similar segments from two given circles; prove that the four tangents at the points of intersection form a parallelogram. (Bryn Mawr, 1894.)

10. Prove that the areas of two triangles which have an angle of the one equal to an angle of the other are to each other as the products of the sides including the equal angles.

Describe an isosceles triangle equal in area to a given triangle and having its vertical angle equal to one angle of the given triangle.

11. Show how to construct the mean proportional to two given lines, proving the construction. Given the sum of two lines, show that their product is greatest when they are equal.

Divide a line of length 10 inches into two parts whose rectangle shall be a maximum. (1895.)

12. How many conditions must be given for the determination of a triangle? If two sides are given, what else must be given? What if two angles are given?

Construct a triangle, given the base and the ratio of the sides.

13. Give and prove the construction for a circle touching the three sides of a triangle.

The centre of the circle inscribed in the triangle ABC is O; the centres of the circles touching one side and the other two produced are A', B', C'; show that the circles on OA', $B'C'$ as diameters meet one another at B, C.

14. Prove that any chord of a circle is bisected by the

diameter perpendicular to the chord. Hence, show that the locus of the bisection of chords through a fixed point is a circle passing through the fixed point and through the centre of the given circle.

Show how to draw a chord of a given circle to pass through a given point and be bisected by a given line. How many chords are there satisfying the given conditions?

15. Prove that the square on the hypothenuse of a right-angled triangle is equal to the sum of the squares on the sides.

The areas of the circles whose diameters are the sides of a certain right-angled triangle are 24 square inches and 72 square inches; find the area of the circle on the hypothenuse as diameter.

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16. In a triangle whose sides are 48, 36, and 50, where do the bisectors of the angles intersect the sides? What are the lengths of the bisectors? (Dartmouth.)

17. In a circle whose radius is 8, what is the length of the arc of a sector of 45°? What is the area of this sector?

18. The sides of a triangle are 5, 6, and 8; find the segments of the last side made by a perpendicular from the opposite angle.

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19. Demonstrate a method of constructing a circle with a given radius which shall be tangent to each of two given right lines. (Johns Hopkins, 1895.)

20. The area of a given equilateral triangle is $36\sqrt{3}$; find the area of the circumscribing circle.

21. Demonstrate a method of constructing a square equivalent to a given triangle.

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22. If LM is one of the non-parallel sides of a trapezoid circumscribed about a circle whose center is O, prove that LOM is a right angle. (Mt. Holyoke, 1895.)

23. The circumference of a circular pond is 86 rods; what is the side of a square containing the same area?

24. Inscribe in a given circle a regular polygon similar to a given regular polygon, and prove.

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25. The area of a rhombus is 240 and its side is 17; find its diagonals. (Pennsylvania, 1895.)

26. Construct a square whose area shall be five times the area of a given square.

27. If the interior bisector of the angle C and the exterior bisector of the angle B of a triangle ABC meet at D, prove that angle $BDC = \frac{1}{2}A$.

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28. Three equal circles touch each other; find the area of the three-pointed figure which they enclose, the radius of each circle being one foot. (Trinity, 1894.)

---

29. The base of an isosceles triangle is a, and the perpendicular let fall from one extremity of the base to the opposite side is b; find the lengths of the equal sides. (Vassar, 1893.)

30. Construct a circle that shall pass through two given points and shall cut from a given circle an arc of given length.

---

31. The four bisectors of the angles of a quadrilateral form a second quadrilateral the opposite angles of which are supplementary. (Wellesley, 1892.)

32. Given a right triangle whose perpendicular sides are a and b, construct the value of x which satisfies the equation $x = \frac{ab}{\sqrt{a^2 + b^2}}$.

33. Find the circumference of a circle whose area is 100.

34. (*a*) On a circle of 4 feet radius, how long is an arc

included between two radii forming an angle of 20 degrees? Prove, deriving the formula employed.

(*b*) Find the area of the regular circumscribed hexagon of a circle whose radius is 1. (1895.)

---

35. If D is the middle point of the side BC of the triangle ABC, and BE and CF are the perpendiculars from B and C to AD, prove that $BE = CF$. (Wesleyan, 1895.)

36. If the apothem of a regular hexagon is 2, find the area of its circumscribed circle.

---

37. If three quantities are proportional, the first is to the third as the square of the first to the square of the second. (Williams.)

STEVENS, 1891.

1. Given a and b, two sides of a $\triangle$; and h, the altitude upon b; to find p, the altitude upon the third side.

2. Two circles intersect in the points P and Q. A secant is drawn through P, cutting the circles at the points A and B, and another through Q, cutting them at C and D. Show that the chords AC and BD, joining the extremities of the secants, are parallel.

3. The bases, one diagonal, and the angle formed by the diagonals being given, to construct the trapezoid.

4. The square of the bisector of the angle of a triangle is equal to the product of the sides containing this angle diminished by the product, etc. Finish and demonstrate the theorem.

5. Given the sides, a, b, and c, of a triangle; to compute the altitude on a in terms of the sides.

6. To construct a circle which shall pass through two given points and touch a given straight line.

HARVARD.

1. The circumference of a circle A passes through the centre of a smaller circle B: prove that their common tangents touch A in points which lie on a straight line tangent to B. (1889.)

2. What is the locus of the middle point of a straight line of given length inscribed in a given circle? The radii of two concentric circles are 36 inches and 39 inches, respectively. A chord of the larger circle touches the smaller; how long is this chord?

3. Two sides of a triangle are 10 inches and 12 inches, respectively, and the angle included between them is 50°. Show how to draw a similar triangle which shall have twice the area of the given triangle.

4. Find the radius and the area of the largest circle that can be cut from a triangular piece of paper every side of which is 4 feet long.

5. A certain equilateral triangle has sides $8\sqrt{3}$ inches long; what is the radius of the circumference circumscribed about this triangle? (1890.)

6. The vertices of a quadrilateral inscribed in a circle divide the circumference into arcs which are to each other as 1, 2, 3, and 4. Find the angles between the opposite sides of the quadrilateral.

7. Prove that the diagonals and the line which joins the middle points of the parallel sides of a trapezoid meet in a point.

8. The radius of a certain circle is 9 inches; find the area of that one of all the regular polygons inscribed in it which has the shortest perimeter. How long a perimeter can a regular polygon inscribed in this circle have?

9. Show how to find on a given indefinitely extended straight line in a plane a point O which shall be equidistant from two given points A, B in the plane. If A and

B lie on a straight line which cuts the given line at an angle of 45° at a point 7 inches distant from A and 17 inches from B, show that OA will be 13 inches. (1892.)

10. Prove that an angle formed by a tangent and a chord drawn through its point of contact is the supplement of any angle inscribed in the segment cut off by the chord. What is the locus of the centre of a circumference of given radius which cuts at right angles a given circumference?

11. Prove that the square described upon the altitude of an equilateral triangle has an area three times as great as that of a square described upon half of one side of the triangle.

12. Find the area included between a circumference of radius 7 and the square inscribed within it.

13. How many lines can be drawn through a given point in a plane so as to form in each case an isosceles triangle with two given lines in the plane? (1893.)

14. Assuming that the areas of two triangles which have an angle of the one equal to an angle of the other are to each other as the products of the sides including the equal angles, prove that the bisector of an angle of a triangle divides the opposite side into parts which are proportional to the sides adjacent to them.

15. A quarter-mile running-track consists of two parallel straight portions joined together at the ends by semi-circumferences. The extreme length of the plot enclosed by the track is 180 yards. Find the cost of sodding this plot at a quarter of a dollar per square yard.

16. A kite-shaped racing-track is formed by a circular arc and two tangents at its extremities. The tangents meet at an angle of 60°. The riders are to go round the track, one on a line close to the inner edge, the other on a line everywhere $5\frac{1}{4}$ feet outside the first line. Show that the second rider is handicapped by about 22 feet. (Sept., 1894.)

HARVARD.

JUNE, 1894.

[In solving problems use for π the approximate value $3\frac{1}{7}$.]

1. Prove that any quadrilateral the opposite sides of which are equal is a parallelogram.

A certain parallelogram inscribed in a circumference has two sides 20 feet in length and two sides 15 feet in length; what are the lengths of the diagonals?

2. Prove that if one acute angle of a triangle is double another the triangle can be divided into two isosceles triangles by a straight line drawn through the vertex of the third angle.

Upon a given base is constructed a triangle one of the base angles of which is double the other. The bisector of the larger base angle meets the opposite side at the point P. Find the locus of P.

3. Show how to find a mean proportional between two given straight lines, but do not prove that your construction is correct.

Prove that if from a point, O, in the base, BC, of a triangle, ABC, straight lines be drawn parallel to the sides, AB, AC, respectively, so as to meet AC in M and AB in N, the area of the triangle AMN is a mean proportional between the areas of the triangles BNO and CMO.

4. Assuming that the areas of two parallelograms which have an angle and a side common and two other sides unequal, but commensurable, are to each other as the unequal sides, prove that the same proportion holds good when these sides have no common measure.

5. Every cross-section of the train-house of a railway-station has the form of a pointed arch made of two circular arcs the centers of which are on the ground. The radius of each arc is equal to the width of the building (210 feet); find the distance across the building measured over the

roof, and show that the area of the cross-section is 3675 $(4\pi - 3\sqrt{3})$ square feet.

HARVARD.

JUNE, 1895.

One question may be omitted.

[In solving problems use for π the approximate value $3\frac{1}{7}$.]

1. Prove that if two straight lines are so cut by a third that corresponding alternate-interior angles are equal the two lines are parallel to each other.

2. Prove that an angle formed by two chords intersecting within a circumference is measured by one-half the sum of the arcs intercepted between its sides and between the sides of its vertical angle.

Two chords which intersect within a certain circumference divide the latter into parts the length of which, taken in order, are as 1, 1, 2, and 5; what angles do the chords make with each other?

3. Through the point of contact of two circles which touch each other externally, any straight line is drawn terminated by the circumferences; show that the tangents at its extremities are parallel to each other.

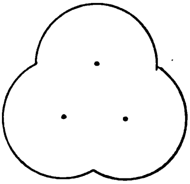
What is the locus of the point of contact of tangents drawn from a fixed point to the different members of a system of concentric circumferences?

4. Prove that if from a point without a circle a secant and a tangent be drawn, the tangent is a mean proportional between the whole secant and the part without the circle.

Show (without proving that your construction is correct) how you would draw a tangent to a circumference from a point without it.

5. Prove that the area of any regular polygon of an even number of sides ($2n$) inscribed in a circle is a mean proportional between the areas of the inscribed and the circum-

scribed polygons of half the number of sides. If n be indefinitely increased what limit or limits do these three areas approach?



6. The perimeter of a certain church window is made up of three equal semi-circumferences, the centers of which form the vertices of an equilateral triangle which has sides $3\frac{1}{2}$ feet long. Find the area of the window and the length of its perimeter.

YALE.

1. From two points, A and B, on an arc of a circle straight lines, AC, AD, BC, BD, are drawn to the ends of the chord of the arc. Find two similar triangles in the figure, prove them similar and write the proportions of their homologous sides. (1889.)

2. If the radius of a circle is 5 inches compute its circumference and its area; also the perimeter, the area and the apothem of an inscribed square. (1890.)

3. Construct an equilateral triangle whose perimeter shall be 12 inches, and inscribe in it a circle. (1891.)

NOTE.—Write nothing in connection with this problem.

4. Prove that if the points of tangency of the above inscribed circle be joined the triangle formed is equilateral, and its perimeter is 6 inches.

5. (*a*) Define a geometrical locus, and give examples of loci from the figure drawn in number 3.

(*b*) Define the limit of a variable, and write the caption of some proposition whose proof depends on the theory of limits.

6. Construct accurately by ruler and compass a parallelogram $ABCD$ having the angle A 45°, the side AB 6 units in length and the altitude 3 of the same units.

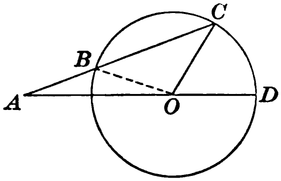
Calculate the length of the side AC. (1892.)

7. Given a straight line AB, of indefinite length, and a point C without it. Find a point in AB equally distant from A and C.

Make the necessary construction accurately with ruler and compass.

In what case is the solution impossible? (1893.)

8. Given an angle COD at the center of a circle and the line CA meeting DO produced in A so that AB is equal to the radius of the circle. Prove that the angle A is equal to one-third of the angle COD.



YALE.

JUNE, 1894.

Geometry (*a*). [*Time, one hour.*]

1. If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

2. To draw a tangent to a given circle so that it shall be parallel to a given straight line.

3. If AB is a chord of a circle, and CE is any chord drawn through the middle point C of the arc AB cutting the chord AB at D, prove that the chord AC is a mean proportional between CD and CE.

4. The areas of two similar triangles are to each other as the squares of any two homologous sides.

5. The area of a circle is equal to one-half the product of its circumference and radius.

Geometry (b). [*Time, 45 minutes.*]

1. What is the number of degrees in each angle of a regular decagon?

2. Find the area in square feet of an equilateral triangle whose side is 3 meters.

3. *ABC* is a right triangle. The sides *AC* and *BC* about the right angle *C* are respectively 50 and 120 feet. Divide the triangle into two parts equal in area by a line *DF* parallel to *BC*. Compute the length of the three sides of the triangle *ADF*.

4. The area of a circle is a hectare. What is its diameter?

5. Calculate in meters the length of a degree on the circumference of the earth, assuming the section of the earth to be a circle whose radius is 3963 miles. [Those taking the preliminary examinations must use logarithms.]

[*For preliminary candidates only.*]

6. Find the value of the following expression by logarithms:

$$\sqrt[3]{\frac{(.06342)^2 \times 187.32}{.34216 \times 6.0372}}.$$

YALE.

June, 1895.

Geometry (a).

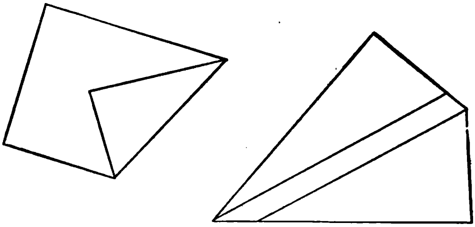
1. (*a*) Define the terms "locus" and "limit of a variable" and give an example of each.

(*b*) Prove that two triangles are similar if their homologous sides are proportional.

(*c*) Through a given point *A* within a circle draw two equal chords.

[*Both the construction (with ruler and compass) and also the proof are required.*]

2. (*a*) Prove that if each of two angles of a quadrilateral is a right angle the bisectors of the other angles are either perpendicular or parallel to each other.



(*b*) Prove that if the radius of a circle is divided in extreme and mean ratio, the greater part is equal to the side of a regular inscribed decagon.

[*The construction is not required.*]

Geometry (b).

One question may be omitted. Logarithmic tables should be used in calculating the answers of two questions.

1. The base of a triangle is 14 inches and its altitude is 7 inches. Find the area of the trapezoid cut off by a line 6 inches from the vertex.

Express the result in square meters.

2. Find the number of feet in an arc of 40° 12′ if the radius of the circle is 0.7539 meters.

3. The length of a chord is 10 feet, and its greatest distance from the subtending arc is 2 feet 7½ inches. Find the radius of the circle.

4. Find the area and the weight in grams of the largest square that can be cut from a circular sheet of tin 16 inches in diameter and weighing 8.2 ounces per square foot.

SHEFFIELD SCIENTIFIC SCHOOL.

1. The perimeter of the circumscribed equilateral triangle is double that of the similar inscribed triangle. (1890.)

2. The area of a certain polygon is S square feet. Find the area of the similar polygon whose perimeter is in the ratio of m to n to that of the given polygon.

4. When the areas of two similar polygons are in the ratio of m to n, in what ratio are the homologous sides? (1893.)

5. In the triangle whose sides are a, b, and c, determine the segments of each side made by the bisector of the opposite angle.

6. Construct a square whose area is three times that of a given square. (1894.)

7. Construct a circle which shall be tangent to a given line and touch a given circle in a given point. (1895.)

SHEFFIELD.

SEPTEMBER, 1894.

Plane Geometry.

[NOTE.—State at the head of your paper what text-book you have studied on the subject and to what extent.]

1. Define equal polygons, equivalent polygons, regular polygons, similar polygons, a maximum polygon, a minimum polygon.

2. If a chord of a circle is greater than another chord, it is nearer the center. State the converse proposition and prove it.

3. The sum of the squares on the four sides of any quadrilateral equals the sum of the squares on the diagonals and four times the square on the line joining middle points of diagonals.

4. Construct a triangle and a square each of which is equivalent to a given pentagon. Give proof of constructions.

5. If the perimeter and base of a triangle are given, when is the area greatest?

SOLID GEOMETRY.

SOLID GEOMETRY.

370. All the propositions of Plane Geometry are based on the assumption that all the points of a figure lie in the same plane, or are *coplanar*. In Solid Geometry this restriction is removed; but as some figures in Solid Geometry are plane while others are not, it follows that some propositions from Plane Geometry can be directly extended to Solid Geometry, while others must be modified or rejected. For example: § 70 is true in space, but § 61 is true only in a plane; § 101 is true in space, but if the polygon of § 118 is replaced by a "polygon" not plane, the theorem does not hold true.

DEF. § 16. A **plane** is a surface such that, whatever two points in it be taken, the straight line through them lies wholly in the surface.

BOOK IV.

LINES AND PLANES IN SPACE.

The postulates and axioms of Plane Geometry are retained. To those are added:

371. POSTULATE III. *A plane can be drawn through any three points.*

372. AXIOM VI. *Through three points not collinear* only one plane can be drawn.*

PROPOSITION I. THEOREM.

373. *A plane is determined by—*

(1) *Three non-collinear points;*
(2) *A line and a point outside it;*
(3) *Two intersecting lines;*
(4) *Two parallel lines.*

[(1) is proved by Ax. VI; (2), (3), and (4) are proved by (1).]

Ex. Show that any theorem in Plane Geometry in regard to a triangle is true in Solid Geometry.

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* Collinear points are points in the same straight line.

PROPOSITION II. THEOREM.

374. *The intersection of two planes is a straight line.*

[Call the intersection of M and N j; let A, B be any two points in j. Prove the straight line AB to be a part of j; $\therefore$ j is straight.]

375. Definitions of **parallel line** and **plane**, and **parallel planes**, are analogous to the definition of parallel lines.

PROPOSITION III. THEOREM.

376. *If a line is parallel to a line in a plane, it is parallel to the plane.*

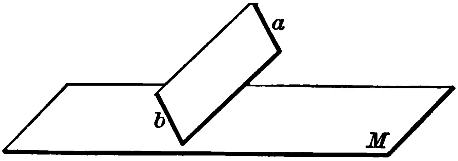

FIG. 146.

[Pass a plane through lines a, b. If a could meet the plane M (say in X), X would be in both planes; *i.e.*, a and b would intersect.] (See Notes, page 218.)

PROPOSITION IV. THEOREM.

377. *If a line is parallel to a plane, it is parallel to the plane's intersection with a plane through the line.*

[Let a be $\parallel$ to M. To prove a parallel to j. Indirect proof.]

378. COR. *If of two parallel lines the first is parallel to a plane and the second passes through a point in the plane, then the second lies in the plane.*

[Prove b and j coincident.]

Ex. If a plane is passed through each of two parallel lines, the intersection of the planes is parallel to each of the lines.

PROPOSITION V. THEOREM.

379. *If each of two lines is parallel to a third line, the two lines are parallel to each other.*

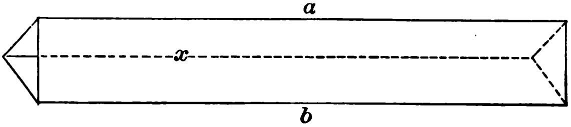

FIG. 147.

[Let a and b be ∥ to x. Call some point on b Q. Pass planes through a, x; b, x; a, Q. Planes b x and a Q intersect in j. Prove b and j coincident.]

(See Note *a*, page 218.)

Ex. If three planes intersect in three lines, the intersections are either concurrent or parallel.

PROPOSITION VI. THEOREM.

380. *If two intersecting lines are parallel respectively to two other intersecting lines, the angles between the first pair are equal to the homologous angles between the second pair.*

[Let a be ∥ to a'; b ∥ to b'; a and b intersect in O; a' and b' in O'. Lay off $OC = O'C'$, $OD = O'D'$. Prove $CDC'D'$ a parallelogram; and OCD, $O'C'D'$ equal triangles.]

(See Note *b*, page 218.)

PROPOSITION VII. THEOREM.

381. *If each of two intersecting lines is parallel to a plane, the plane of these lines is parallel to the first plane.*

[Let a and b be parallel to M. Call the (impossible) intersection of M and plane ab x.]

Ex. Parallel lines included between parallel planes are equal.

PROPOSITION A. PROBLEM.

382. *To construct through a given point a plane parallel to a given plane.*

383. PROBLEM. *To construct through a given line a plane parallel to a given line.*

384. PROBLEM. *To construct through a given point a plane parallel to each of two given lines.*

385. SCH. Show in every case when only one parallel can be drawn.

PROPOSITION VIII. THEOREM.

386. *The intersections of two parallel planes by a third plane are parallel to each other.*

387. THEOREM. *If two lines are cut by three parallel planes, the corresponding segments are proportional.*

388. SCH. Extend § 387 to any number of lines; also to any number of planes.

389. DEF. A line is **normal** to a plane when it is perpendicular to every line drawn through its foot in the plane.

A plane is **normal** to a line when the line is normal to the plane.

PROPOSITION IX. THEOREM.

390. *If two lines are perpendicular to a given line at one point, their plane is normal to the given line.*

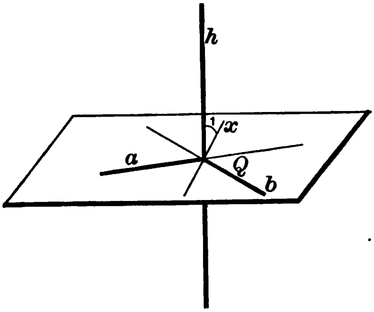

FIG. 148.

[Let a, b be ⊥ to h at Q. Let x be any line in plane ab through Q. To prove x ⊥ to h.

Lay off equal distances on h from Q. Prove that the two adjacent angles on h made by x ($\angle$s x_1, x_2) are equal by showing that they occur in equal triangles.]

PROPOSITION B. PROBLEM.

391. *To construct a plane normal to a line at a given point in the line.*

392. THEOREM. *The locus of the perpendiculars to a line at one point in the line is the plane normal to the line at that point.*

PROPOSITION C. PROBLEM.

393. *To construct a line normal to a plane at a given point in the plane.*

[Let Q be the point in plane M. Draw through Q in M two lines a, b. Construct plane A' normal to a; B' normal to b at Q. Prove j (intersection of A' and B') normal to M.]

394. PROBLEM. *To construct a plane normal to a line from a point outside the line.*

395. SCH. Show in every case when only one perpendicular or normal is possible.

EXS. **1. What is the locus of points equidistant from the extremities of a line?**

2. What is the locus of points equidistant from the circumference of a circle?

PROPOSITION X. THEOREM.

396. *If one of two parallel lines is normal to a plane, the other is also normal to the plane.*

[Use Prop. VI.]

397. COR. *Two lines normal to the same plane are parallel to each other.*

PROPOSITION XI. THEOREM.

398. *If a line is normal to one of two parallel planes, it is normal to the other plane also.*

399. COR. *Two planes normal to the same line are parallel to each other.*

PROPOSITION D. PROBLEM.

400. *To construct a line normal to a plane from a given point outside the plane.*

[Use Props. A, C and XI.]

PROPOSITION XII. THEOREM.

401. *The normal is the shortest line from a point to a plane.*

402. COR. (1) *If two obliques from a point to a plane are equal, they cut off equal distances from the foot of the normal.*

(2) *Analogs of* §§ 82, 84, 85.

403. DEF. The **distance** from a point to a plane is the length of the normal from the point to the plane.

EX. *Two parallel planes are everywhere equally distant.*

404. THEO. *The locus of the foot of an oblique of fixed length from a given point to a given plane is a circle whose center is the foot of the normal from the point to the plane.*

405. DEFS. A **dihedral angle** (or a **dihedral**) is the figure formed by two planes which meet in a line.

The **edge** is the line of intersection.

The **faces** are the planes.

Many terms referring to plane angles are transferred to the analogous dihedrals.

406. DEF. A **plane angle of a dihedral** is an angle formed by drawing perpendiculars to the edge at a point in the edge—one in each face.

407. THEO. *All plane angles of the same dihedral (or of equal dihedrals) are equal.*

408. SCH. Any plane angle of a dihedral is called *the* plane angle of the dihedral.

409. THEO. *Two dihedrals are equal if their plane angles are equal.*

PROPOSITION XIII. THEOREM.

410. *Any two dihedrals are proportional to their plane angles.*

411. SCH. A dihedral is *measured* by its plane angle.

PROPOSITION XIV. THEOREM.

412. *If a line is normal to a plane, any plane through the line is perpendicular to that plane.*

[Prove the plane angle a right angle.]

Ex. To construct a right dihedral with a given line in a given plane as the edge.

413. COR. 1. *If a line in one face of a right dihedral is perpendicular to the edge, it is normal to the other face.*

414. COR. 2. *If a line is normal to one face of a right dihedral from any point in the other face, it lies in that other face.*

415. THEOREM. *If each of two intersecting planes is perpendicular to a third plane, their intersection is normal to the third plane.*

416. THEOREM. *A plane normal to the edge of a dihedral is perpendicular to each of the faces.*

417. Theorems and definitions in dihedrals analogous to

(1) §§ 60–67.
(2) § 71.
(3) §§ 95–100.
(4) §§ 119, 120.
(5) § 121.

PROPOSITION E. PROBLEM.

418. *To pass a plane through a given line perpendicular to a given plane.*

When will there be more than one plane perpendicular to the given plane?

419. DEFS. The **projection of a point on a plane** is the foot of the normal from the point to the plane.

The **projection of a line or curve on a plane** is the locus of the projections of its points on the plane.

420. THEOREM. *The projection of a straight line on a plane is a straight line.*

[Use Problem E and definition § 419.]

421. THEOREM. *The angle between a line and its projection is less than the angle between the line and any other line drawn through its foot in the plane.*

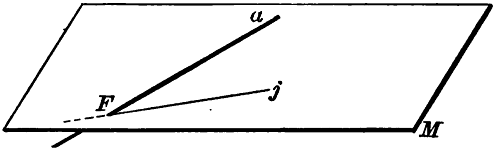

FIG. 149.

[Let j be the projection of a on M. Let b be any line in M drawn through the foot of a,—F.

From any point P in a draw the normal PD, and on b lay off $FG = FD$. Draw PG. Prove angle $ab > aj$ by § 79.]

422. DEF. The **angle between a line and a plane** is the angle between the line and its projection on the plane.

PROPOSITION F. PROBLEM.

423. *To draw a common perpendicular to two non-coplanar lines.*

[Pass a plane *M* through *b* parallel to *a*. Project *a* on *M*. Call the projection *j*, and call the intersection of *j* and *b*, *C*. Then *CD*, ⊥ to *M*, is the perpendicular required.]

424. COR. 1. *There is only one common perpendicular to two non-coplanar lines.*

425. COR. 2. *The common perpendicular to two non-coplanar lines is the shortest line joining the two lines.*

426. DEF. The **distance** between two non-coplanar lines is the length of the common perpendicular between them.

427. SCHOLIUM. While two non-coplanar lines cannot intersect (and so cannot form an angle), yet they are sometimes regarded as having an angle equal to the angle between two intersecting lines each of which is parallel to one of the given lines. This angle of the non-coplanar lines will equal the angle formed by their projections on a plane constructed from problem § 384.

It follows from this that a normal to a plane is perpendicular to every line in the plane.

EXERCISES. 1. Extensions of exs. 4, 5, page 42.

2. If from a point within a dihedral normals to the faces are drawn, the angle between the normals is the supplement of the plane angle of the dihedral.

3. What is the locus of points equally distant from *A*, *B*, *C* (any three points)?

4. Find the point (*O*) equally distant from *A*, *B*, *C*, *D*. (*D* is not coplanar with *A*, *B*, *C*.)

5. Find the point (*P*) equally distant from the lines *AB*, *BC*, *CD*, *DA*.

428. DEFS. A **polyhedral angle** (or a **polyhedral**) is the figure formed by three or more planes which meet at a common point.

The **faces** are the planes. The **edges** are the intersections of the planes. The **vertex** is the common point of the faces, and also of the edges.

Polyhedrals are distinguished as **trihedrals, tetrahedrals,** etc., as the number of faces (and edges) is 3, 4, etc.

Dupuis calls a polyhedral a *corner;* a trihedral is then a three-faced corner.

The **face-angles** are the angles at the vertex between the consecutive edges.

If a plane cuts all the edges, the polygon of section is called *the base* of the polyhedral. When the base is a convex polygon the polyhedral is called convex, and only convex polyhedrals will be considered.

429. DEF. If the edges (and faces) are extended through the vertex, there is formed a polyhedral called the **vertical** or **opposite** of the first figure. Evidently all the face-angles of the first figure are equal to homologous parts in the second, and the same is true of the dihedral angles; i.e., all the elements of the first polyhedral are equal to corresponding parts in its vertical, and the sequence of parts is the same, but if they are selected clockwise in one they occur counter-clockwise in the other. The two polyhedrals will therefore be called **inversely equal.** The term *symmetric,* or *conjugate,* is often applied to them. It should be noticed that two such polyhedrals are vertical only when placed in a certain opposite or vertical position, but they are inversely equal (or symmetric) when in any position.

429*a*. Two vertical polyhedrals are inversely equal.

429*b*. Two inversely equal polyhedrals may be placed opposite.

PROPOSITION XV. THEOREM.

430. *The sum of any two angles of a trihedral is greater than the third.*

Let the trihedral *O-ABC* have angle *AOB* $(= \angle O_1)$ the largest angle.

It is to be proved that $\angle BOC + \angle COA > O_1$.

In face *1* draw $\angle AOH = \angle COA$ $(= \angle O_3)$.

On *OC* and *OH* lay off the equal parts *OF*, *OG* respectively.

Through any point, *P*, on *OA* pass the plane *PFG* cutting *OB* in *K*.

Then $$PF = PG;$$

$$\therefore \quad FK > GK;$$

$$\therefore \angle FOK > \angle GOK;$$

$$\therefore O_3 + O_2 > O_1.$$

PROPOSITION XVI. THEOREM.

431. *The sum of all the face-angles of a polyhedral is less than four right angles.*

Let *O-ABC*... be the polyhedral whose base is *ABC*... From any point (*P*) within the polygon *ABC*... draw *PA*, *PB*, etc.

Denote these base triangles by I, II, etc., and the triangles *OAB*, etc., by (1), etc.

$$\Sigma \text{ angles in I} = \Sigma \text{ angles in (1)}.$$

$$\Sigma \text{ base } \angle\text{s in I} < \Sigma \text{ base } \angle\text{s in (1)}.$$

$$\therefore \Sigma \angle\text{s } P_{\mathrm{I}} > \Sigma \angle\text{s } O_1.$$

$$\therefore O_1 + O_2 + \text{etc.} < 4 \text{ right angles.}$$

$$(\Sigma O_1 < 2\pi.)$$

432. THEOREM. *If two trihedrals have the face-angles of one equal respectively to the face-angles of the other, the two trihedrals are equal either directly or inversely.*

Let $O_1 = O'_1$, etc.

Lay off $OA = O'A' = OB = O'B' = OC = O'C'$. On OA and $O'A'$ lay off $OP = O'P'$. At P and P' pass planes normal to the edges intersecting the bases in FG, $F'G'$. Prove angles FPG, $F'P'G'$ equal.

433. SCH. A trihedral—or its symmetric—is determined by three face-angles. If the direction of the angles about the vertex is given, the trihedral is completely determined.

What other conditions determine a trihedral?

EXERCISES.

1. If from any point within a trihedral normals be drawn to the faces, the angles between the normals are supplements of the corresponding dihedrals.

2. In ex. 1 call the vertex of the trihedral O, and the intersection of the normals R. Pass planes through the consecutive normals forming the trihedral R. Then the dihedrals of R are supplements of the corresponding face-angles of O.

DEF. The trihedrals R and O are sometimes called *reciprocals.*

3. What is the locus of points equidistant from the faces of a trihedral? (See § 417, (5).) Make the theorem about "bisectors."

4. What is the locus of points equidistant from the edges of a trihedral?

5. The sum of the dihedrals of a trihedral is less than six right angles. ($\Sigma \angle$s $OA < 3\pi$.)

6. The sum of the dihedrals of a trihedral is greater than two right angles. ($\Sigma \angle$s $OA > \pi$.)

DEF. If AB, BC, CD, DA are non-coplanar, the figure $ABCD$ is called a *skew-quadrilateral* or a *gauche-quadrilateral.*

7. If the mid-points of the consecutive sides of a skew-quadrilateral be joined, the figure formed is a parallelogram.

8. In ex. 7 if the quadrilateral is $ABCD$ and the mid-points are E, F, G, H, then EG and FH bisect each other. Make the theorem.

9. If L and M are the mid-points of the diagonals in ex. 7, then EG, FH and LM are concurrent.

10. In ex. 9 LM is bisected.

11. If $AB = BC = CD = DA$, the diagonals are perpendicular.

12. In a trihedral the planes through the bisector of a face-angle and the opposite edge have a common line.

13. x and y are non-coplanar lines, A any point on x, B any point on y; find the locus of the mid-point of AB.

14. Find a point at distance l from points A, B, and plane M.

15. Find a point in a given plane and at distance l from each of two other given planes.

16. Find the locus of points in a given plane and at a given distance l from a given point outside the plane.

17. Find the locus of points in a given plane and at a given distance l from a given line.

18. If two face-angles of a trihedral are equal, the dihedrals opposite those face-angles are equal.

19. Converse of ex. 18.

20. If two face-angles of a trihedral are right angles, the dihedrals opposite those face-angles are right dihedrals.

21. Converse of Ex. 20.

DEF. The trihedral of ex. 20 is a *bi-rectangular* trihedral.

22. If the three face-angles of a trihedral are right angles, the three dihedrals are right dihedrals. Prove also the converse.

DEF. The trihedral of ex. 22 is *tri-rectangular*.

23. Through any point on one edge of a tetrahedral to pass a plane whose section with the faces is a parallelogram.

BOOK V.

POLYHEDRONS. THE CYLINDER AND THE CONE.

434. Defs. A **prismatic surface** is a figure formed by planes whose intersections are parallel.

The **faces, edges** and **dihedrals** of a prismatic surface have obvious meanings.

A **section** of a prismatic surface is the figure formed by a plane cutting all the edges.

A **right section** is a section normal to the edges.

435. Defs. A **polyhedron** is a closed figure, or a solid, bounded by planes. The **edges** and **vertices** of the polyhedron are the edges and vertices of its polyhedrals. The **faces** of the polyhedron are the plane polygons which bound it.

A **diagonal** of a polyhedron is a line joining two vertices not in the same face. It is sometimes called a solid diagonal.

The faces sometimes mean the planes indefinitely extended beyond the edges.

A polyhedron is *convex* when no face if extended will enter the polyhedron; in this case any section of the polyhedron is a convex polygon. Polyhedrons are assumed to be convex unless it is otherwise stated.

436. Def. A **prism** is a polyhedron formed from a prismatic surface by two parallel sections. The **lateral faces** and **lateral edges** of a prism are the faces and edges of the prismatic surface. The **bases** of a prism are the parallel sections of the prismatic surface. The **altitude** (or height) of a prism is the perpendicular distance between its bases.

Prisms are described as *triangular*, *hexagonal*, etc., from the analogous plane figures.

437. COR. 1. *The bases of a prism are equal.*

438. COR. 2. *The lateral faces of a prism are parallelograms.*

439. COR. 3. *The lateral edges of a prism are equal.*

440. DEFS. A **right prism** is a prism whose lateral edges are normal to the bases.

An **oblique prism** is a prism whose lateral edges are oblique to the bases.

441. COR. *The lateral faces of a right prism are rectangles.*

442. DEF. A **regular prism** is a right prism whose bases are regular polygons.

443. COR. *The lateral faces of a regular prism are equal rectangles.*

PROPOSITION I. THEOREM.

444. *If two prisms have a trihedral of one directly equal to a trihedral of the other, and the faces about the first trihedral equal to the corresponding faces about the second trihedral, the prisms are equal.*

[Prove by superposition.]

445. COR. 1. *Two right prisms are equal when they have equal bases and equal altitudes.*

[Show what to do when the trihedrals are inversely equal.]

446. DEF. A **truncated prism** is a polyhedron formed from a prismatic surface by two sections not parallel.

447. COR. 2. *If two truncated prisms have a trihedral of one equal, etc.* [Statement and proof as in § 444.]

448. Def. The **volume** of a solid is the amount of space contained in it.

449. Def. **Equivalent** solids are solids having the same volume.

Proposition II. Theorem.

450. *If two prisms are formed from the same prismatic surface and have equal edges, they are equivalent.*

[Use § 447.]

451. Cor. *If a right prism and an oblique prism have the base of the first equal to a right section of the second, and the altitude of the first equal to a lateral edge of the second, the two prisms are equal.*

452. Def. A **parallelepiped** is a prism whose bases are parallelograms.

453. Theorem. *The opposite faces of a parallelepiped are equal and parallel.*

454. Sch. Any face of a parallelepiped may be chosen as a base.

Exercises. 1. To construct a parallelepiped when a trihedral and the lengths of its three edges are given.

2. Two parallelepipeds are equal when a trihedral in one is directly equal to a trihedral in the other, and the three edges of the first trihedral are equal respectively to the three edges of the second.

455. Theorem. *The diagonals of a parallelepiped bisect each other.*

456. Theorem. *The diagonals of a parallelepiped are concurrent.*

457. Def. The **center** of a parallelepiped is the mid-point of any diagonal.

458. Def. *An axis* of a parallelepiped is the line joining the centers of two opposite faces.

459. Theorem. *The axes of a parallelepiped are concurrent at the center.*

EXERCISES. 1. Each axis is parallel to four edges.

2. The angles between the axes are equal to the face-angles of the trihedrals of the parallelepiped.

3. How many face-angles of a parallelepiped can be acute?

4. What is the relation of the trihedrals which are diagonally opposite?

5. A line through the center of a parallelepiped and intercepted by the opposite faces is bisected at the center.

6. How many unequal acute dihedrals can occur in a parallelepiped?

460. SCH. Observe that the arrangement of acute or obtuse face-angles at one vertex of a parallelepiped determines their arrangement throughout the figure and gives a certain *character* to the parallelepiped. A similar remark may be made of the dihedrals of any trihedral.

Notice also that at one trihedral—

(1) There may be one face-angle right and all the dihedrals may be oblique;

(2) There may be one right dihedral and all the face-angles oblique;

(3) If two face-angles are right, there are two right dihedrals, and conversely;

(4) See ex. 22, p. 176.

461. DEFS. A **right parallelepiped** is already defined by §§ 440, 452.

A **cuboid** * is a parallelepiped whose bases and faces are rectangles.

All the dihedrals of a cuboid are right dihedrals (§ 460 (4)).

The **dimensions** of a cuboid are the three edges at any vertex.

462. DEF. A **cube** is a cuboid all of whose edges are equal.

Hence its faces are equal squares.

EXERCISES. 1. The diagonals of a cuboid are equal.

2. What is the square on the diagonal of a cuboid equal to?

3. What is the square on the diagonal of a cube equal to?

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* Also called a *rectangular parallelepiped*. The term *cuboid* was suggested by Mr. R. B. Hayward.

PROPOSITION III. THEOREM.

463. *The plane through two diagonally opposite edges of a parallelepiped divides it into two equivalent triangular prisms.*

[Call the two triangular prisms T', T''. Construct in the same prismatic surface an equivalent right parallelepiped (§ 450) composed of triangular prisms T_1, T_2.

Then $T' \Leftrightarrow T_1 \Leftrightarrow T_2 \Leftrightarrow T''$.]

464. PROBLEM. *To construct a cuboid equivalent to a given parallelepiped.*

[Let a, b, c be the edges of the parallelepiped at the base vertex V. Construct the equivalent right parallelepiped P' in the prismatic surface whose edge is a. Call its edges at V a, b', c'.

Construct the equivalent cuboid D in the prismatic surface whose edge is b'.]

NOTE. If the plane ab is regarded as the base plane, the given parallelepiped and the required cuboid have the same height, and a pair of corresponding edges equal.

EXERCISES. 1. If two parallelepipeds have equal bases and equal heights, they are equivalent.

2. If two triangular prisms have equal bases and equal heights, the prisms are equivalent.

3. To construct a triangular prism equivalent to a given prism. (Use method of § 309.)

4. To construct a parallelepiped (and then a cuboid) equivalent to a given prism.

PROPOSITION IV. THEOREM.

465. *Two cuboids with equal bases are proportional to their altitudes.*

[Proof as in § 334.]

465a. COR. *Two cuboids having two dimensions common are proportional to the third dimensions.*

PROPOSITION V. THEOREM.

466. *Two cuboids with equal altitudes are proportional to their bases.*

Let P and P' have bases bc, $b'c'$, and altitude $a = a'$; to prove $P : P' ::$ base bc : base $b'c'$.

Construct cuboid Q with dimensions a, b, c'.

Then $\quad P : Q ::$ etc., $\quad$ or $\quad P = \frac{c}{c'} \cdot Q$;

and $\quad Q : P' :: b : b'$, $\quad$ or $\quad Q = \frac{b}{b'} \cdot P'$.

$$\therefore P = \frac{b \cdot c}{b' \cdot c'} P';$$

$$\therefore P : P' :: bc : b'c'.$$

In the above outline what two meanings do a, b, etc., have? What does "base bc" mean? What does the "product $b \cdot c$" mean? Why can one expression replace the other?

NOTE. A cube whose edge is a linear unit is generally taken as a unit of volume.

PROPOSITION VI. THEOREM.

467. *The number of cubical units in any cuboid is equal to the product of the numbers of linear units in its dimensions.*

[For method of proof see § 349.]

467a. *Restatement (see* § 350). *The volume of a cuboid is equal to the product of its base and altitude.*

$$\text{Vol.} = abc.$$

468. COR. 1. *A parallelepiped is equal to the product of its base and altitude.*

469. COR. 2. *A triangular prism is equal to the product of its base and altitude.*

470. COR. 3. *Any prism is equal to the product of its base and altitude.*

$$\text{Vol.} = Bh.$$

What other product is a prism equal to?

471. COR. 4. *Two prisms with equivalent bases and equal altitudes are equivalent.*

Exs. Find the entire surface and the volume of these prisms:

1. Regular triangular, base edge = 5, lateral edge = 8.
2. Regular hexagonal, base edge = 2, lateral edge = 10.
3. Right quadrangular, base edges a, $\sqrt{6}$, height = 5. Angle between two base edges = 30°.
4. Regular pentagonal, area of base = 30, height = 9.

472. DEFS. A **pyramid** is a polyhedron bounded by the faces of a polyhedral and a plane cutting all the edges of the polyhedral.

The **base** is the polygon of section.

The **vertex, lateral faces, lateral edges, base edges, altitude** and **lateral surface** are terms whose meanings are obvious.

473. THEOREM. *Any section of a pyramid through the vertex is a triangle.*

474. THEOREM. *Any section of a pyramid parallel to the base is similar to the base.*

475. COR. 1. *The ratio of similitude of two parallel sections of a pyramid is equal to the ratio of their distances from the vertex.*

476. COR. 2. *The areas of two parallel sections of a pyramid are proportional to the squares of their distances from the vertex.*

477. DEFS. A set of prisms is **circumscribed** about a pyramid when the prisms wholly include the pyramid and have for their lower bases the base of the pyramid and sections parallel to that base.

A set of prisms is **inscribed** in a pyramid when the prisms are wholly within the pyramid and have for their upper bases sections parallel to the base of the pyramid.

PROPOSITION VII. THEOREM.

478. *If a set of prisms of equal altitude is circumscribed about a pyramid, the pyramid is the limit of the sum of the prisms when the number of prisms is indefinitely increased.*

Divide the altitude of the pryamid (P) into any number of equal parts, n, and through the points of division pass planes parallel to the base. On the base and on these sections construct prisms circumscribing the pyramid.

Call one of these prisms C.

Also on these same sections construct prisms inscribed in the pyramid.

Call one of these prisms I.

It is to be proved that ΣC approaches P as a limit when the number of parts (n) into which the altitude is divided is indefinitely increased.

Excepting the lowest circumscribed prism, every circumscribed prism is equivalent to the inscribed prism just below it. [?

$\therefore \Sigma C - \Sigma I =$ the lowest circumscribed prism.

$= C_1$.

When n is indefinitely increased C_1 is indefinitely diminished.

$\therefore \Sigma C - \Sigma I$ approaches 0 as a limit when n is indefinitely increased.

But $P > \Sigma I$ and is $< \Sigma C$;

$\therefore \Sigma C - P$ approaches 0 as a limit when n is indefinitely increased.

$\therefore \Sigma C$ approaches P as a limit when n is indefinitely increased.

479. COR. *If a set of prisms of equal altitude is inscribed in a pyramid, the pyramid is the limit of the sum of the prisms when the number of prisms is indefinitely increased.*

Proposition VIII. Theorem.

480. *Two pyramids having equivalent bases and equal altitudes are equivalent.*

To prove $P \Leftrightarrow P'$.

Place the pyramids with their bases in the same plane and divide the altitudes into equal parts by planes parallel to the base plane. Construct the sets (C, C') of circumscribed prisms. $\Sigma C \Leftrightarrow \Sigma C'$.

$\therefore P \Leftrightarrow P'$.

Proposition IX. Theorem.

481. *The volume of a triangular pyramid is equal to one-third the volume of a triangular prism of the same base and altitude.*

[Let the triangular pyramid be $O.ABC$.

With base ABC and edge BO construct the prism P having the lateral edges AH, BO, CM. Call the pyramid $OABC$ T_1; pass a plane through OAM; call the pyramid $OACM$ T_2; call the pyramid $AHOM$ T_3. Prove $T_1 \Leftrightarrow T_2$, and $T_2 \Leftrightarrow T_3$. $\therefore T_1 \Leftrightarrow \frac{1}{3}P$.]

482. Cor. 1. *A triangular pyramid is equal to one-third the product of its base and altitude.*

483. Cor. 2. *Any pyramid is equal to one-third the product of its base and altitude.*

$$Vol. = \tfrac{1}{3}Bh.$$

484. Theorem. *A truncated triangular prism is equivalent to the sum of three pyramids whose common base is the base of the prism and whose vertices are the three vertices of the section.*

[Prove $T_2 \Leftrightarrow E.ABC$, and $T_3 \Leftrightarrow F.ABC$.]

$$Vol. = \tfrac{1}{3}(h_1 + h_2 + h_3)B.$$

485. DEF. A **frustum** of a **pyramid** is the part of a pyramid between the base and a section parallel to the base.

PROPOSITION X. THEOREM.

486. *A frustum of a triangular pyramid is equivalent to the sum of three pyramids whose common altitude is the altitude of the frustum and whose bases are the lower base, the upper base, and the mean proportional between the bases of the frustum.*

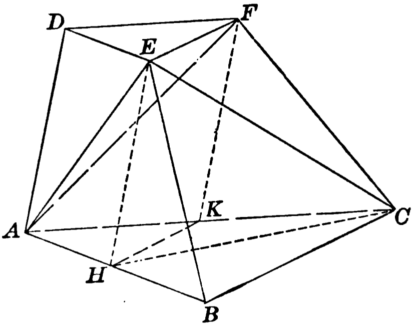

FIG. 150.

Pass a plane through EF parallel to AD. Draw HC. Prove $E.AFC \Bumpeq H.AFC = F.AHC$. Use Ex. 2, p. 129; then frustum $\Bumpeq$ sum of three pyramids, etc.

487. COR. *A frustum of any pyramid is equivalent to the sum of three pyramids, etc.* (as in § 496).

Construct a triangular pyramid equivalent to the pyramid containing the given frustum. Prove the two frustums equivalent remainders.

$$Vol. = \tfrac{1}{3}h(B + b + \sqrt{Bb}).$$

488. DEF. A **regular pyramid** is a pyramid whose base is a regular polygon and the projection of whose vertex on the base is the center of the base.

489. COR. *The lateral faces of a regular pyramid are equal isosceles triangles.*

490. DEF. The **slant height** of a regular pyramid is the altitude of a lateral face.

EXS. Find entire surface and volume of the following:

1. Regular triangular pyramid, base edge = 2, lateral edge = 5.

2. Regular quadrangular pyramid, base edge = 3, altitude = 5.

3. Regular hexagonal pyramid, base edge = 4, lateral surface = 120.

4. Triangular pyramid, every edge = 6.

5. Triangular pyramid, all edges equal, entire surface = 24.

6. A regular quadrangular frustum has base edges 25 and 9, and lateral edge 12. Find the entire surface and volume.

7. A regular triangular frustum has base edges 9 and 4, and the lateral edge makes an angle of 60° with the lower base. Find entire surface and volume.

EXERCISES.

NOTE. A tetrahedron (or *four-faced*) is a triangular pyramid. A *regular* tetrahedron has all four triangles equal.

1. The plane which bisects a dihedral of a tetrahedron divides the opposite edge into segments proportional to the adjacent faces.

2. In a tetrahedron the planes through any three concurrent edges and the middle points of the opposite edges intersect in a line.

3. In a tetrahedron the planes through the edges and the middle points of the opposite edges intersect in a point.

(The point is the center of gravity of the tetrahedron.)

4. Prove that the point of ex. 3 divides the line of ex. 2 into segments in the ratio 3 · 1.

5. The planes through the three lateral edges of a tetrahedron perpendicular to the opposite faces intersect in a line. (Let the base of the tetrahedron be normal to one edge.)

6. Pass a plane through a tetrahedron so that the section shall be a parallelogram.

7. The sum of the normals from any point within a regular tetrahedron to the faces is constant. The sum equals what ?

8. In a tetrahedron the lines joining the mid-points of opposite edges intersect in a point. (These lines are sometimes called diameters.)

9. The point of ex. 8 is the center of gravity of the tetrahedron.

10. The diameters of a tetrahedron bisect one another.

11. If two tetrahedrons have a trihedral of one equal to a trihedral of the other, they are to each other as the products of the three edges of the equal trihedrals.

12. In a tetrahedron the four pyramids having for bases the faces of the tetrahedron and for a common vertex the center of gravity of the tetrahedron are equivalent.

13. The opposite edges of a regular tetrahedron are perpendicular to each other.

14. The base of any pyramid is less than the sum of the faces.

15. In a pyramid whose base is a polygon of n sides, (1) what is the number of edges? (2) What is the number of faces? (3) What is the number of vertices?

How does the number of edges in a pyramid compare with the sum of the numbers of faces and vertices?

16. Show by a method similar to that of ex. 15 how the number of edges in a prism compares with the sum of the numbers of faces and vertices.

491. THEOREM. *The number of edges in any polyhedron is equal to the number of faces plus the number of vertices less two.*

Let e denote the number of edges;
f " the number of faces;
v " the number of vertices.

It is to be proved that in a polyhedron

$$e = f + v - 2.$$

Suppose the polyhedron formed by beginning with a single face (I), and putting face after face into place, but so that the "nest" of faces shall always form a close surface (i.e., we are not to build around a missing face).

When any face is added (excepting the last one, which completes the polyhedron) the number of new edges is one more than the number of new vertices.

With face I, $e = f + v - 1.$

Add face II, and the number of new *faces and vertices* equals the number of new edges.

∴ with faces I and II $e = f + v - 1.$

Also with three faces $e = f + v - 1,$ and so on.

When the last face is added,
e is not increased;
v is not increased;
f is increased *one*.

$\therefore$ at last $$e = f + v - 2.$$

NOTE. This theorem is commonly called Euler's Theorem of Polyhedrons.

EXERCISES.

1. A right triangular prism has for base edges 6, 8, 10, and for height 15. What is the entire surface of the pyramid T_2 of § 481.

2. In the figure of ex. 1 a plane bisects a lateral edge and is parallel to the base. Find the volume of the frustum of T_1.

3. Find the entire surface of the frustum of ex. 2.

4. A regular triangular prism with base edge equal to 1 is truncated so that its lateral edges are 9, 8, 3. Find its volume.

5. A cuboid whose base edges are 15, 20 is truncated so that three lateral edges are 24, 21, 20. Find the volume.

6. A regular square pyramid has base edge 5 and lateral edge 14. What is its volume and entire surface?

7. A regular triangular pyramid has a base 5 sq. in. and height 10 in. Find the entire surface.

8. The lateral faces of a regular pyramid are five equilateral triangles with one inch for a side. Find the entire surface and the volume of the pyramid.

9. How many solid regular tetrahedrons, edge $= 1$ inch, can be molded from a plate of glass 16 in. by 9 in., and $\frac{1}{4}$ in. thick?

10. A grain-hopper is a frustum of a square pyramid; sides of bases 4 ft. and $1\frac{1}{4}$ ft.; edge, between bases, 2 ft. How many bushels will the hopper hold?

THE CYLINDER AND THE CONE.

492. If a line always passing through the perimeter of a polygon is always parallel to a line not in the plane of the polygon, the locus is evidently a prismatic surface. If the perimeter of the polygon is replaced by any curve, the resulting locus is a *cylindrical surface.* The curve is called the director. Just as the only curve treated in elementary plane geometry is the circle, so the only cylindrical surface much regarded in elementary solid geometry is the circular cylindrical surface; its director is a circle.

Similarly, a pyramidal surface may be conceived as the locus of a line always passing through the vertex and moving about the perimeter of a polygon. When the polygon is replaced by a curve the locus is a *conical surface;* and the curve considered here will be a circle.

493. DEFS. A **generator** is a point, line, or plane figure which moves so that its locus is a figure of the next higher dimension (§ 6).

A **director** is a line (straight, broken, or curved) which a generator always intersects.

The director is viewed as guiding the generator.

494. DEFS. A **curved surface** is a surface no portion of which is plane.

A **cylindrical surface** is the locus of a line moving so as to be always parallel to its first position. The surface is **circular** when the director is circular.

495. Def. A **circular cylinder** is a solid contained by a circular cylindrical surface and two parallel planes, one of which contains the director.

496. Defs. A **right circular cylinder** is a circular cylinder whose generator is normal to the plane of the director.

A right circular cylinder is also called a **cylinder of revolution.** If a rectangle is revolved about one side as an axis, the resulting locus is a right circular cylinder.

497. Defs. A **circular cone** and a **right circular cone** may be defined in a similar way.

A **cone of revolution** is also analogous to a cylinder of revolution.

498. Def. An **element** of a cylinder or cone is the generator of the curved surface in any position.

499. Theorem. *The section of a cylinder through an element is a parallelogram.*

500. Cor. *The section of a cone through an element is a triangle.*

501. Def. A **plane** is **tangent** to a cylinder (or a cone) when the plane passes through an element of the curved surface without intersecting that surface.

502. Problem. *To construct a plane tangent to a cylinder* (*or a cone*) *through a given element of the surface.*

[Draw a line tangent to the base at the foot of the element.]

When is a *line* tangent to a cylinder or cone?

503. Theorem. *A prism with a regular polygon for base may be inscribed within* (*or circumscribed about*) *a circular cylinder.*

504. Cor. *A pyramid with a regular polygon for base may be inscribed within* (*or circumscribed about*) *a circular cone.*

505. THEOREM. *A circular cylinder is the limit of an inscribed (or circumscribed) prism with a regular polygon for base when the number of sides of the polygon is indefinitely increased.*

[A rigorous proof of this theorem is based on axioms analogous to Axiom II and Axiom V, § 351, of plane geometry.]

506. COR. *A circular cone is the limit of an inscribed (or circumscribed) pyramid with a regular polygon for base when the number of sides of the polygon is indefinitely increased.*

507. SCH. 1. Consistently with this view of the cylinder and cone as the respective limits of the prism and pyramid with regular base, many definitions and formulas of this Book are extended to cylinders and cones. The student should make these extensions.

508. SCH. 2. If the generator is extended through the vertex of a conical surface a double conical surface is obtained, one on each side of the vertex. These are called the two *nappes* of the surface.

EXERCISES. Find the entire surface and volume—

1. Of a right circular cylinder, radius of base = 7, altitude = 10.
2. Of a right circular cone, radius of base = 5, slant height = 12.
3. Of a right circular frustum, radii of bases 6 and 24, altitude = 20.

Note. In the following formulas the prism is right, the pyramid and its frustum are "regular" figures, and the cylinder, the cone, and its frustum are figures of revolution.

FORMULAS.

Lat. surf. of prism $= p.h.$
" " " cylinder $= c.h = 2\pi r.h.$
" " " pyramid $= \frac{p.l}{2}.$
" " " cone $= \frac{c.l}{2} = \pi r.l.$
" " " frustum of pyr. $= \frac{p+p'}{2}.l.$
" " " frustum of cone $= \frac{c+c'}{2}.l = \pi(r+r')l.$
Vol. of prism $= b.h.$
" " cylinder $= b.h = \pi r^2 h.$
" " pyramid $= \frac{b.h}{3}.$
" " cone $= \frac{b.h}{3} = \frac{\pi r^2 h}{3}.$
" " frustum of pyr. $= \frac{b+b'+\sqrt{bb'}}{3}.h.$
" " frustum of cone $= \frac{b+b'+\sqrt{bb'}}{3}.h$
$= \pi\frac{r^2+r'^2+rr'}{3}.h.$

THE REGULAR POLYHEDRONS.

509. DEF. A **regular polyhedron** is a polyhedron whose faces are all equal regular polygons, and whose dihedrals are all equal.

It follows that all the face-angles are equal, and all the polyhedrals are equal.

510. 1. Three equilateral triangles can be combined to form a polyhedral. This is the polyhedral of the regular tetrahedron.

2. Four equilateral triangles can be combined to form a polyhedral of a regular octahedron.

3. Five equilateral triangles can be combined to form a polyhedral of a regular icosahedron.

More than five equilateral triangles cannot be combined to form a polyhedral.

4. Three squares can be combined to form a trihedral of a regular hexahedron.

More than three squares cannot form a polyhedral.

5. Three regular pentagons can be combined to form a polyhedral of a regular dodecahedron.

More than three regular pentagons cannot be combined to form a polyhedral; and no regular polygons of more than five sides can be used to form a polyhedral.

It is not *proved* here that these five figures can be constructed, but it is evident that there cannot be *more* than five regular polyhedrons. (See Note 14, page 223.)

511. SCH. If lines are drawn joining the center of each face to the centers of the adjacent faces, these lines will form the edges of another polyhedron called *the conjugate* of the first.

The conjugate of a tetrahedron is a tetrahedron.

The conjugate of a hexahedron is an octahedron.

The conjugate of a dodecahedron is an icosahedron.

SIMILAR POLYHEDRONS.

512. THEOREM. *If four lines are in proportion, the cubes on those lines are in proportion.*

513. DEF. Two **polyhedrons** are **similar** if they are bounded by the same number of faces, similar in pairs and similarly placed, and have their homologous dihedrals equal.

514. THEOREM. *Two similar polyhedrons may be divided into the same number of tetrahedrons, similar in pairs and similarly placed.*

[Beginning at a vertex A (and its homolog A') divide all the faces, except those forming the polyhedral A, into triangles, and draw lines from A forming tetrahedrons having a common vertex A. Prove the tetrahedrons similar.]

515. THEOREM. *If two polyhedrons are similar, two homologous lines are in the ratio of similitude.*

COR. *If two polyhedrons are similar, two homologous faces are proportional to the squares on homologous lines.*

EXERCISE. *Two tetrahedrons having a trihedral in one equal to a trihedral in the other are proportional to the products of the including edges.*

516. THEOREM. *Two similar tetrahedrons are proportional to the cubes of homologous edges.*

517. COR. *Two similar polyhedrons are proportional to the cubes of homologous lines.*

518. SCH. *Similar cylinders* and *similar cones* are easily defined, and analogous properties established. The most important cases are the figures of revolution; and these are similar when the generating rectangles, or generating right triangles, are similar.

EXERCISES. 1. A cylinder is 40 inches in diameter and 8 ft. long. When it rests on an element the water in it is 10 in. deep. How deep will it be when the cylinder is placed on end?

2. A cone is 3 inches high, radius of base $= 2$ in. What is the angle of the sector which the lateral surface develops into?

3. A two-quart pail, in shape a frustum of a cone, has diameters of bottom and top 4 and 6 inches. How high is it? Where on the side is the one-quart mark?

4. An hour-glass is 6 inches high and has a diameter of 2 inches. It contains 2 cu. in. of sand. Where (marked on an "element" of the cone) is the upper surface of sand at the beginning of an hour? Where at the end of an hour? And where are the surfaces when the sand has half run through?

BOOK VI.

THE SPHERE.

519. DEF. A **sphere** is a solid bounded by a surface all the points of which are equally distant from a point within called the center.

The sphere is a closed figure.

520. COR. *A sphere is the locus of a semicircle revolved about its diameter as axis.*

PROPOSITION I. THEOREM.

521. *The section of a sphere by a plane is a circle.*

[The section is the locus of a point on the semicircumference. Prove by definition of circle.]

522. DEF. The **poles** of a circle are the extremities of the sphere's diameter which is normal to the plane of the circle.

523. DEFS. A **great circle** of a sphere is a section by a plane through the sphere's center.

A **small circle** is a section by a plane not through the center.

524. COR. 1. *On a sphere all great circles are equal.*

525. COR. 2. *On a sphere any two great circles bisect each other.*

526. THEOREM. *If two planes equally distant from the center of a sphere intersect the sphere, the small circles are equal.*

527. COR. *If two small circles of a sphere are equal, their planes are equally distant from the center.*

EXERCISE. State and prove the analogs of §§ 162, 163.

528. THEOREM. *The intersection of two spheres is a circle.*

[Prove as in Prop. I.]

529. DEF. A plane (or a line) is **tangent** to a sphere when it touches the sphere in only one point even when extended.

PROPOSITION A. PROBLEM.

530. *To construct a plane tangent to a sphere at a given point on the sphere.*

531. COR. *A line in a tangent plane through the point of contact is tangent to the sphere.*

PROPOSITION B. PROBLEM.

532. *To circumscribe a sphere about a given tetrahedron.*

[See Ex. 4, page 172.]

533. COR. *Through any four points not coplanar a spherical surface can be passed.*

PROPOSITION C. PROBLEM.

534. *To inscribe a sphere in a given tetrahedron.*

EX. To escribe a sphere to a given tetrahedron and tangent to only one given face externally.

535. SCH. It is easy to see from the symmetry of the figures that there may be a sphere inscribed within (or circumscribed about) any regular polyhedron; also that there may be a sphere to which the edges of a regular polyhedron will be tangent.

536. DEF. If a plane is passed through the center of a sphere and through two points on the surface not collinear with the center, the intersection will be a great circle through the two points on the surface. This great circle is analogous to the straight line joining two points in plane geometry. It is sometimes called a *spherical line.*

537. Def. The angle between two intersecting great circles, or a **spherical angle,** is the angle between the straight lines tangent to the circles at the point of intersection.

In general the angle between two intersecting curves is the angle between their tangents at the point of intersection.

Of course the line tangent to a circle must be in the same plane as the circle.

538. *If a trihedral be placed with its vertex at the center of a sphere, the faces will intersect the surface in three arcs, and the dihedrals will form three spherical angles.*

The three intercepted arcs form a spherical triangle.

Also, (1) the arcs will measure the face-angles of the trihedral, and (2) the spherical angles will measure the dihedrals of the trihedral. Therefore many of the properties of a trihedral may be simply restated as properties of the spherical triangle.

539. If the faces of the trihedral are extended through the center, the inversely equal trihedral will be formed; and its faces will intersect the sphere in a spherical triangle whose sides and angles are equal to homologous parts in the first spherical triangle, but the parts will be arranged in the inverse order. The two spherical triangles will be called **inversely equal.**

They are also called *symmetric,* and *opposite.* See § 429.

540. Theorem. *If two inversely equal spherical triangles are isosceles, they are directly equal.*

(Prove by superposition.)

541. Theorem. *If two spherical triangles are inversely equal, they are equivalent.*

[Pass a plane through the vertices of each triangle and construct the pole O, O' of each small circle. Then use the equations

$$\text{Sph.}\ \triangle\ OAB = \text{sph.}\ \triangle\ O'A'B', \text{ etc.}]$$

542. Theorem. *The sum of the sides of a spherical triangle is less than the circumference of a great circle.*

Ex. The sum of the sides of a spherical polygon is less than the circumference of a great circle.

543. DEF. The **distance** between two points on a sphere is the length of the minor arc (or the spherical line) joining the points.

544. THEOREM. *All the points of a spherical circle are equally distant from a pole of the circle.*

545. DEF. The **polar distance** of a spherical circle is the distance from any point in the circle to the nearer pole.

546. SCH. The polar distance of a great circle is a quadrant of a great circle; this is usually called simply a quadrant.

547. THEOREM. *If a point is a quadrant's distance from two given points on a sphere (not collinear with the center), then it is the pole of the great circle through those points.*

548. THEOREM. *A spherical angle is measured by the intercepted arc of the great circle whose pole is the vertex of the angle.*

PROPOSITION II. THEOREM.

549. *If the vertices of a spherical triangle are the poles of the sides of a second triangle, then the vertices of the second are poles of the sides of the first.*

[Call the triangles ABC and $A'B'C'$, and let A' be the pole of the side BC, etc. Use § 547.]

550. DEF. Two spherical triangles are **polar** when the vertices of one triangle are poles of the sides of the other triangle.

SCH. Either triangle may be chosen as the *first*, and the other is then its polar. With a given triangle T there are four triangles any one of which satisfies literally the definition of polar just given. *The polar* which is selected is that one in which the pole (A') of a given side (BC) lies on the same side of its arc (BC) as does the homologous vertex (A).

PROPOSITION III. THEOREM.

551. *In two polar triangles each angle of one triangle is measured by the supplement of the side lying opposite in the other triangle.*

[Take angle A.

$$\angle A = DE = B'E + DC' - B'C' = 180° - B'C'.]$$

PROPOSITION IV. THEOREM.

552. *The sum of the angles of a spherical triangle is more than two right angles and less than six right angles.*

[Use the sum of the angles + sum of polar's sides. Then subtract sum of polar's sides.]

553. DEF. The **spherical excess** of a spherical triangle is the excess of the sum over its angles of two right angles.

Similarly the spherical excess of a spherical polygon is the excess of the sum of its angles over the sum of the angles of a plane polygon of the same number of sides.

EXERCISES. 1. Prove that the sum of the angles of a sph. pentagon is more than six right angles.

2. Can the sum of the sides of a spherical triangle be greater than the sum of the angles? In a spherical triangle of which every side is a quadrant how do the sums compare?

554. DEF. A **lune** is the portion of the surface of a sphere bounded by two semi-circumferences of great circles.

555. COR. 1. *The two angles of a lune are equal.*

556. COR. 2. *On the same sphere, if two lunes have equal angles, the lunes are equal.*

557. COR. 3. *On the same sphere two lunes are proportional to their angles.*

558. COR. 4. *A lune is to the surface of a sphere as its angle is to four right angles.*

NOTE. The spherical surface is a lune of 360°, and a hemispherical surface is a lune of 180°.

PROPOSITION V. THEOREM.

559. *The area of a spherical triangle is equal to one half a lune whose angle is the spherical excess of the triangle.*

Denote the spherical triangle ABC by T, and denote $A'BC$ by T_1, $B'CA$ by T_2, $C'AB$ by T_3, and $A'B'C$ by Q.

Then T_3 and Q are equivalent; [?

$$T + T_1 = \text{lune } A,$$

$$T + T_2 = \text{lune } B,$$

$$T + Q = \text{lune } C.$$

$$\therefore 2T + T + T_1 + T_2 + Q = \text{lune } A + \text{lune } B + \text{lune } C.$$

$$\therefore 2T + \text{hemisphere} = \text{lune } A + \text{lune } B + \text{lune } C.$$

$$\therefore 2T = \text{lune } A + \text{lune } B + \text{lune } C - \text{hemisphere}.$$

$$\therefore 2T = \text{lune } (A + B + C) - \text{hemisphere}$$

$$= \text{lune } (A + B + C - 180^\circ)$$

$= $ lune whose angle is the spherical excess of the triangle ABC.

$$\therefore T = \tfrac{1}{2} \text{ lune whose angle, etc.}$$

560. SCH. There are two units often used in measuring spherical triangles:

(1) The trirectangular triangle, which equals one-eighth of the spherical surface.

(2) The spherical degree, which is a birectangular triangle whose vertical angle is one degree. Evidently 90 sph. degrees = 1 trirectangular triangle.

The spherical degree must be carefully noted as a *surface* measure, not an angular measure.

EXERCISE. On a sphere a triangle is drawn whose angles are $A = 110^\circ$, $B = 85^\circ$, $C = 60^\circ$. Find the area of the triangle. (Sph. ex. $= 75^\circ$. Area $= \frac{1}{2}$ lune $75^\circ = 75$ sph. degrees $= \frac{75}{90}$ trirectangular triangle. It is convenient to view the half-lune as a half formed by dividing the lune into two birectangular triangles.

SPHERICAL SURFACE GEOMETRY.

561. Some of the analogies of Spherical Surface Geometry to Plane Geometry are very obvious, and it is instructive to notice those definitions and axioms which mark the difference between the two geometries. Regarding the spherical line (§ 536) as analogous to the straight line in the plane, Axiom I is in general true, but not universally true, on the sphere, because between two diametrically opposite points any number of spherical lines can be drawn.

Axiom III has no analog in Spherical Surface Geometry and parallel spherical lines do not occur. Consequently the theorems depending on parallels do not apply to a spherical surface, and there are no "similar polygons" on a sphere.

Prop. VII of Book I must be modified—

(1) because in general two perpendiculars can be drawn on a sphere from a point outside a spherical line to that line;

(2) because from the pole of the spherical line any number of perpendiculars can be drawn to the line.

Prop. IX of Book I must be correspondingly modified.

The fundamental theorems of Book I relating to equal and unequal triangles may be extended to spherical triangles, but should be carefully stated. Thus Prop. IV becomes: "If two spherical triangles are mutually equilateral, they are equal either directly or inversely."

The following Propositions from Book I should be restated and proved:

I, II, III, IV, X, XI, XII.

NOTE. Instead of assuming an analog of Axiom II it is sometimes *proved* in Spherical Surface Geometry that the spherical line joining two points on a sphere is the shortest line which can be drawn on the sphere between the points. As circles are the only curves strictly belonging in elementary geometry, there is substituted the following

562. THEOREM. *The spherical line joining two points is less than any small-circle arc on a sphere joining the same points.*

[For it may be proved in *plane* geometry that a minor arc joining two points is less than any arc of shorter radius joining the same two points.]

563. THEOREM. *If two spherical triangles are mutually equiangular, they are equal either directly or inversely.*

[Prove the polars of the given triangles equal, and then pass back to the original triangles.]

MEASUREMENT OF THE SPHERE.

564. THEOREM. *The lateral surface of a frustum of revolution is equal to the product of the altitude of the frustum multiplied by the circumference of a circle whose radius is the segment of the right bisector of an element between the element and axis of revolution.*

It is to be proved that the locus of AB is equal to

$$\overline{DC}.\,2\,\pi\,\overline{HK}.$$

Locus of AB is equal to

$$\overline{AB}.2\pi\overline{HI}$$
$$=2\pi\overline{AB}.\overline{HI}$$
$$=2\pi\overline{AF}.\overline{HK}= \text{etc.}$$

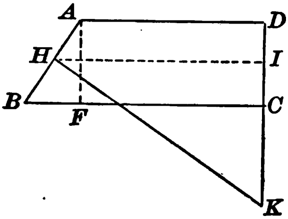

FIG. 151.

NOTE. The theorem is general and includes the cases when AD is zero and when $AD = BC$.

565. DEFS. A **zone** is the portion of the surface of a sphere included between two circles whose planes are parallel.

The **altitude** of a zone is the distance between the planes of its circles.

Limiting cases are when one circle is zero (or one plane tangent) giving a zone of one base, and when both planes are tangent (altitude equal to diameter).

A zone may be regarded as the locus of an arc revolved about a diameter as axis. The altitude of the zone is then the projection of the generator on the axis.

PROPOSITION VI. THEOREM.

566. *The area of a zone is equal to the product of the altitude of the zone multiplied by the circumference of a great circle.*

[Divide the arc into any number of equal parts. The area of the zone will be the limit of the sum of the areas generated by the chords when the number of chords is indefinitely increased.]

$$(Z = 2\pi ra.)$$

567. COR. *The area of the surface of a sphere is equal to four great circles.*

$$(S = 4\pi r^2.)$$

EXERCISES. 1. On a sphere the areas of two zones are proportional to their altitudes.

2. *If a right cylinder is circumscribed about a sphere, the area of a zone is equal to the lateral surface of the cylinder contained between the planes of the zone.*

3. *Extend ex. 2 to comparing the whole spherical surface and the whole lateral surface of the cylinder.*

4. Compare the surface of a sphere and the entire surface of the circumscribed right cylinder.

5. *The surfaces of two spheres are proportional to the squares of their radii.*

6. *The area of a zone of one base is equal to the area of a circle whose radius is the chord of the generating arc of the zone.*

568. DEF. A **spherical segment** is the solid bounded by a zone and the planes of the zone.

Limiting cases are a segment of one base, and the sphere.

The generator of a spherical segment is one half the difference of two segments of a circle when the chords of the segments are parallel.

569. Def. A **spherical sector** is the locus of a circular sector revolved about a diameter (not intersecting the sector) as an axis.

Limiting cases are a spherical cone, and the sphere.

570. Def. A **spherical wedge** is the solid bounded by a lune and its two semicircles.

A limiting case is the sphere.

571. A **spherical pyramid** is the solid bounded by a spherical polygon and the planes of the sides of the polygon.

572. Theorem. *If an isosceles triangle is revolved about an axis which is in its plane, passes through its vertex, but does not intersect the triangle, the solid generated is equivalent to the product of the area generated by the base multiplied by one-third the altitude of the triangle.*

Within the surface *EL* draw the lateral surface of a regular inscribed frustum of a pyramid, and on its lateral faces as bases construct pyramids having a common vertex at *O*. The volume of the sum of these pyramids is equal to the product of the sum of the bases multiplied by one-third their common altitude.

Increase the number of bases, and the limit of the sum of the pyramids is the locus of the triangle,

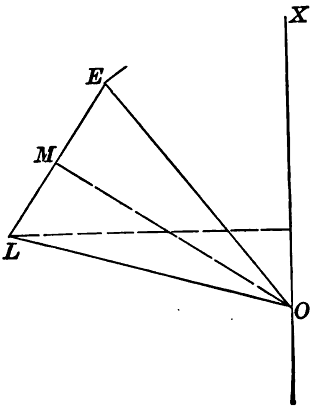

Fig. 152.

the limit of the sum of the bases is the locus of *EL*,
the limit of the altitude of the pyramids is *OM*.

PROPOSITION VII. THEOREM.

573. *The volume of a spherical sector is equal to the product of its zone multiplied by one third the radius of the sphere.*

Let $OEAL$ be the circular sector (Fig. 152.) Divide EAL into any number of equal parts, and draw the equal chords EF, FH, etc. The volume generated by $OEFHKL$ is equal to locus of $EFHKL \times \frac{1}{3}$ altitude of EOF. Increase the number of equal parts into which EL is divided.

$$\left(\text{Sph. sector} = Z \times \frac{r}{3}\right).$$

574. COR. *The volume of a sphere is equal to the product of the area of the surface multiplied by one third the radius.*

$$\begin{aligned}(\text{Sph. vol.} &= \frac{4}{3}\pi r^3 \\ &= \frac{1}{6}\pi D^3.)\end{aligned}$$

575. SCH. 1. The truth of the cor. § 574 is obvious by observing that if a sphere is circumscribed by a polyhedron the volume of the polyhedron is equal to the area of its surface multiplied by one-third the radius. If planes are drawn tangent to the sphere at points inclosed by the polyhedron, vertices of the polyhedron may be sliced off making a smaller polyhedron. Evidently the polyhedrons "approach" the sphere, although this does not furnish a rigorous proof that the sphere is the limit of such polyhedrons.

576. SCH. 2. In the same way it becomes evident that the volume of a spherical pyramid is equal to the product of the area of its base multiplied by one-third the sphere's radius.

577. THEOREM. *On the same sphere two spherical wedges are proportional to their angles.*

578. COR. *The volume of a spherical wedge is equal to the area of its lune multiplied by one-third the radius of the sphere.*

579. PROBLEM. *To find the volume of a spherical segment when the radius of the sphere and the distances of the bases from the center are given.*

The spherical segment may be obtained from the corresponding sector by adding (or subtracting) cones. As the distances from the center are given, the radii of the bases of the cones and the altitude of the sector are easily found.

580. SCH. If the radii of the bases of a spherical segment are r' and r'' and the altitude is h, then

$$\text{Vol. sph. seg.} = \tfrac{1}{2}\pi r'^2 h + \tfrac{1}{2}\pi r''^2 h + \tfrac{1}{6}\pi h^3.$$

581. SCH. All spheres are regarded as similar figures; and similar solids—such as similar spherical pyramids, similar spherical cones, etc.—are cut from spheres of unequal radii.

FORMULAS.

Surface of sph. △ = sph. excess.
" " zone $= 2\pi r . h.$
" " sphere $= 4\pi r^2.$

Vol. of sphere $= \dfrac{4}{3}\pi r^3.$
" " sph. pyr. $= \dfrac{b . r}{3}.$
" " sph. segment $= \dfrac{\pi r'^2 h + \pi r''^2 h + \frac{1}{3}\pi h^3}{2}.$

EXERCISES.

1. A section of a hemisphere parallel to the base bisects the volume. What is the ratio of the surfaces of the two segments?

2. A spherical gold bead is $\frac{1}{32}$ of an inch thick, and the outside diameter is $\frac{1}{4}$ of an inch. How large a solid cube can be made from it?

3. In a cylinder of revolution there are inscribed a sphere and a cone of revolution. Compare the three volumes.

4. In a sphere there are inscribed a cylinder of revolution whose axial section is a square, and a cone of revolution whose axial section is an equilateral triangle. Compare the three volumes.

5. Compare the surfaces in ex. 3.

6. The cone of ex. 4 is called an equilateral cone. When its conical surface is developed, what is the angle of the sector?

7. The four angles of a certain district of the earth's surface were found to be 100°, 150°, 85°, 95°. If the earth's radius is 3960 miles, what is the area of this district?

8. A thimble-shaped solid has for radius of lower base 8 inches, radius of upper base 6 inches, radius of spherical cap 5 inches. What is the whole volume?

9. The frustum of a right pyramid has for bases rectangles, the lower 3×9 inches, the upper 1×3 inches. The altitude is $7\frac{7}{11}$ inches What is the volume? What is the lateral surface?

10. Compare the entire surfaces, and also the volumes, of a sphere, the equilateral circumscribed cone, and the equilateral circumscribed cylinder.

FIGURES OF REVOLUTION.

582. THEOREM. *If a triangle is revolved about an axis which is in its plane and does not intersect the triangle, the volume generated is equal to the area of the triangle multiplied by the circumference of a circle whose radius is the distance from the centroid of the triangle to the axis.*

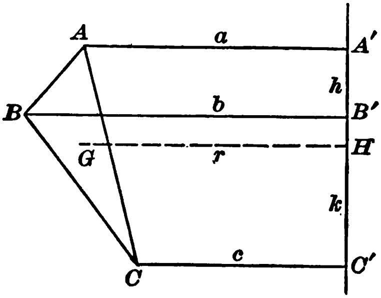

FIG. 153.

Let ABC be the triangle, $A'C'$ the axis, G the intersection of the medians (or the centroid). Let AA' $(=a)$, BB' $(=b)$, CC' $(=c)$, GH $(=r)$ be perpendiculars to $A'C'$. Let $A'B'=h$, $B'C'=k$.

It is to be proved that vol. $ABC=$ area $ABC \cdot 2\pi r$.

Case I. When area $ABC=$ sum of two trapezoids less a third trapezoid.

$$\text{Vol. } A'B = \pi \cdot \frac{h}{3}(a^2+b^2+ab); \qquad (1)$$

$$\text{Vol. } BC' = \pi \cdot \frac{k}{3}(b^2+c^2+bc); \qquad (2)$$

$$\text{Vol. } AC' = \pi \cdot \frac{h+k}{3}(a^2+c^2+ac). \qquad (3)$$

Add equations (1) and (2) and subtract (3):

$$\text{Vol. } ABC = \frac{\pi}{3}(b^2h + abh + b^2k + bck - a^2k - c^2h - ach - ack) \qquad (4)$$

$$= \pi\frac{(a+b+c)}{3}(bh + bk - ch - ak) \qquad (5)$$

$$= \pi r\,[(a+b)\,h + (b+c)\,k - (a+c)\,(h+k)] \qquad (6)$$ [Ex. 3, p. 105.

$$= 2\pi r\left[\frac{a+b}{2}h + \frac{b+c}{2}k - \frac{a+c}{2}(h+k)\right] \qquad (7)$$

$$= 2\pi r\,.\,\text{area of triangle } ABC. \qquad (8)$$

CASE II. When the area of the triangle is equal to one trapezoid less the sum of two trapezoids.

The equation corresponding to (4) will factor in a similar way, giving

$$\text{Vol. } ABC = 2\pi r\left[\frac{a+c}{2}(h+k) - \frac{a+b}{2}h - \frac{b+c}{2}k\right] \qquad (7')$$

$$= 2\pi r\,.\,\text{area of triangle } ABC. \qquad (8)$$

583. COR. *If a triangle is revolved about an axis in its plane and passing through its vertex, the volume generated is equal to the area generated by the base multiplied by one-third the altitude.*

If $c = 0$, equation (5) becomes

$$\text{Vol. } ABC = \frac{\pi}{3}(a+b)(bh + bk - ak)$$

$$= \frac{2\pi}{3}(a+b)\left[\frac{a+b}{2}h + \frac{b}{2}k - \frac{a}{2}(h+k)\right]$$

$$= \frac{2\pi}{3}(a+b)\left[\frac{\overline{AB}\,.\,p}{2}\right]$$

$$= \pi(a+b)\,\overline{AB}\cdot\frac{p}{3}.$$

This furnishes an independent proof of a theorem which includes § 572.

584. SCH. Special cases in § 582 are $a = c$; $a = c = 0$; $h = 0$.

NOTE. If the locus of the centroid is regarded as the director, the theorem of § 582 may be simplified in its statement.

585. DEFS. An *isosceles* trapezoid is a trapezoid whose nonparallel sides are equal.

The *median* of a trapezoid is the line joining the mid-points of the bases.

586. THEOREM. *If an isosceles trapezoid is revolved about an axis which is in its plane, does not intersect the trapezoid and is parallel to the median, the volume generated is equal to the area of the trapezoid multiplied by the circumference of the circle whose radius is the distance from median to axis.*

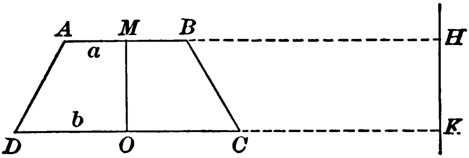

FIG. 154.

Let $ABCD$ be the trapezoid, HK the axis, MO the median.

Call $AM = a$, $DO = b$, HM (or KO) $= r$, $HK = d$.

$$\text{Vol. } HD = \pi\,\frac{d}{3}(\overline{HA}^2 + \overline{KD}^2 + \overline{HA}\,.\,\overline{KD}) \qquad (1)$$

$$\text{Vol. } BK = \pi\,\frac{d}{3}(\overline{HB}^2 + \overline{KC}^2 + \overline{HB}\,.\,\overline{KC}) \qquad (2)$$

$$\text{Vol. } BD = \pi\,\frac{d}{3}(\overline{HA}^2 - \overline{HB}^2 + \overline{KD}^2 - \overline{KC}^2 + \overline{HA}\,.\,\overline{KD} - \overline{HB}\,.\,\overline{KC}) \quad (3)$$

$$= \pi\,\frac{d}{3}[(HA + HB)\,(HA - HB) + \text{etc.}$$

$$= \pi\,\frac{d}{3}[(2r\,.\,2a + 2r\,.\,2b + (r + a)\,(r + b) - (r - a)\,(r - b)]$$

$$= \pi\,\frac{d}{3}[6ar + 6br]$$

$$= 2\pi r d(a + b)$$

$$= 2\pi r\,.\,\frac{AB + CD}{2}\,d$$

$$= \text{circum. generated by } O \times \text{area trapezoid.}$$

587. COR. *The volume generated by a regular polygon revolving about an exterior axis which is in its plane and which is parallel to an axis of symmetry is equal to the area of the polygon multiplied by the circumference whose radius is the distance between the two axes.*

588. SCH. The corollary will apply to any rectilinear figure which has symmetry about an axis.

589. DEF. An **anchor-ring** is the figure generated by revolving a circle about an axis which is in its plane and does not intersect the circle.

The generator is the circle which is revolved.

The director is the locus of the center of the generator.

THEOREM.

590. *The volume of an anchor-ring is equal to the area of the generator multiplied by the length of the director.*

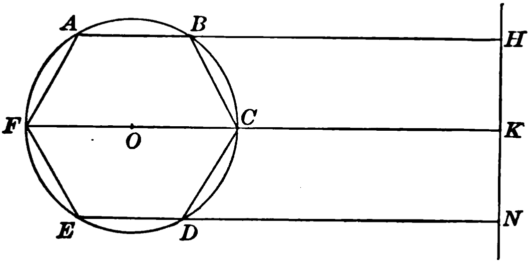

FIG 155.

Let the circle with center O be revolved about the axis HN.

Call the radius of the generator a, and the radius of the director r.

It is to be proved that the volume of the anchor-ring is equal to $\pi a^2 . 2\pi r$, or $2\pi^2 a^2 r$.

Inscribe in the generator any regular polygon having an axis of symmetry parallel to the axis of revolution.

Increase indefinitely (by doubling) the number of sides of the polygon.

Vol. polygon $= 2\pi r$. area polygon.
Vol. circle $= 2\pi r$. area circle.
Vol. anchor-ring $= 2\pi r . \pi a^2 = 2\pi^2 a^2 r$.

591. THEOREM. *The area of the surface generated by a regular polygon revolving about an exterior axis which is in its plane and is parallel to an axis of symmetry is equal to the perimeter of the polygon multiplied by the director.*

In Fig. 154 it is easily seen that the convex surface generated by AD and BC is equal to $2\pi r . (\overline{AD} + \overline{BC})$.

592. THEOREM. *The area of the surface of an anchor-ring is equal to the circumference of the generator multiplied by the director.*

Surface anchor-ring $= 2\pi a \,.\, 2\pi r = 4\pi^2 ar$.

NOTE. The theorems for the surface and the volumes of the anchor-ring are special cases of two theorems discovered by Pappus and often called Guldin's Theorems.

See Dupuis's "Synthetic Solid Geometry," and article "Mensuration" in *Encyc. Brit.*

EXERCISES. 1. A rubber tire in shape of an anchor-ring has greatest diam. 51 inches, inner diam. 47 inches. Find its surface and volume.

2. What is the diameter of the generator of an anchor-ring in which the number of cubic units in the volume equals the number of square units in the surface? How about the radius of the director?

3. The radius and the altitude of a cylinder is $2a$. Compare its volume with the volume of the largest anchor-ring which can be turned from it.

4. An anchor-ring whose radii a and r are 2 and 5 is bisected and moulded into two similar anchor-rings. What are the radii of the small rings?

MISCELLANEOUS EXERCISES.

1. In a cube whose edge is 3 feet a sphere and a cone of revolution are inscribed; find the volume and the entire superficial area of each of the three solids. (Cornell.)

2. The diagonal of a cube circumscribed about a sphere is 9 ft. in length; by how much do the area and the volume of the cube exceed the area and the volume of the sphere?

3. Find the locus of those points of a plane at which a given straight line, not lying in the plane, subtends a right angle. Show when the locus becomes a point and when it disappears.

4. Given e, the length of the edge of a regular tetrahedron, find an expression for the volume of the tetrahedron, and one for the volume of the circumscribed sphere.

5. The radius of a sphere is 7 ft. The volume of a certain seg-

ment of one base is one half the volume of the spherical sector of which it forms a part. How many cubic feet does it contain? (Harvard, 1890.)

6. The area of a lune whose angle is 30° is $37\frac{5}{7}$ sq. ft. Find the volume of the sphere. Find also the altitude of a zone which shall have the same area as the lune. (1891.)

7. The number of cubic feet in a certain sphere is equal to the number of square feet in its surface. Find its radius. (1892.)

8. Show that if a man were at a diameter's distance above the surface of the earth he could see one third of its surface.

9. A cylindrical box is to be made which will just hold seven pills. Which is the cheaper shape for it, gun-barrel or grindstone? How much cheaper? How much more would it cost to line the box than to coat the pills? (1893.)

10. A can buoy is made by cutting a circular sheet of iron two meters in diameter into two semicircles, bending each half into a conical surface, and fastening these surfaces together base to base. How great a weight must be attached to the buoy to submerge it? Given that a cubic decimeter of water weighs a kilogram.

11. How much tin will it take to make a milk-pail if the diameter of the base is one foot, the diameter of the top is 16 inches, and the slant height is 14 inches? How much milk will the pail hold? (1894.)

12. The angles of a spherical pentagon are 90°, 120°, 100°, 140°, and 150°; the radius of the sphere is 7 inches. Find the area of the pentagon in square inches. (1895.)

13. A ranchman 6 feet 7 inches tall, standing on a level plain, agrees to buy, at $7 an acre, all the land in sight. How much must he pay? Given 640 acres make a square mile.

14. Suppose a sphere of radius 5; upon a section of the sphere, at a distance 3 from the center, as a base, a cone is drawn whose elements touch the sphere; find the surface and volume of the cone. (Sheffield, 1892.)

15. A regular triangular pyramid has a for its altitude and each side of its base. Find the area of a section parallel to the base and distant $\frac{1}{3}a$ from the vertex, and get the volume of the original pyramid. (1893.)

16. A sphere circumscribes a cube whose edge is 5 inches; get the surface and volume of the sphere, and compare them with the surface and volume, respectively, of the cube. (1894.)

17. An equilateral triangle revolves about one of its altitudes. If altitude is $3h$, find the volumes and total surfaces of the solids gener-

ated by the triangle, the inscribed circle, and the circumscribed circle. (1895.)

18. A sphere of radius 10 inches is cut by a plane at a distance of 6 inches from the center; (*a*) prove that the section made is a circle; (*b*) find the area and volume of the cone whose base is this circle and whose vertex is the center of the sphere; (*c*) find the ratio of the areas of the two portions into which the spherical surface is divided by the plane. (1897.)

19. A valve consists of a cone (diameter of base 3 inches, slant height 3 inches) and, resting in this, a ball. When the ball is just submerged there is one inch between the level of the water and the top of the valve (or base of the cone). How much water will the valve contain?

20. A saucer in the shape of a segment of a sphere of one base is 6 inches wide and 1 inch deep. What is the radius of a hemispherical cup of the same volume?

21. A cone of revolution whose height is 10 inches and the radius of whose base is 5 inches has a round hole an inch in diameter bored through it; what is the volume of the wood removed? (Harvard.)

22. The sides of an equilateral spherical triangle are 11 inches long. Find the number of square inches in its polar triangle if the radius of the sphere is 14 inches. (1897.)

23. If the radius of the earth is 4000 miles and the height of the atmosphere is 50 miles, find the number of cubic miles in the atmosphere.

NOTES ON MODELS FOR SOLID GEOMETRY.

Solid geometry is reduced almost to the simplicity of plane geometry when models instead of plane drawings are studied.

Very satisfactory models can be made from cardboard for nearly all the figures of solid geometry, and the making of these models is a most instructive part of the study. The following suggestions are by no means an exhaustive list of the possible devices, and the pupil will enjoy ingenious contrivances of his own which illustrate especially some obscure or difficult part of the figure.

It should be borne in mind that these models are not the "intellectual figments" of the demonstrations, and therefore in making the model any order of steps of construction is permissible that results in the desired model.

Cardboard of about the thickness of an ordinary visiting-card will give good results.

Pasteboard boxes can also be utilized for some figures.

A very few skeleton drawings have been given in the text as illustrations of a sort of drawing the pupil must learn to interpret and to produce. The *plane-ness* of them will soon disappear if the pupil will use his imagination on the models. Fortunately the earlier figures are the easiest to model and to imagine and to draw.

I. In Book IV.

a. Two intersecting planes. Pass one card through a slit in the other; or slit both cards and join them as in a "lock-joint." A small brace—in the shape of a triangular prism—will hold them firm at any desired angle.

b. Two planes can be fixed parallel by using a dihedral – such as the edge of a pasteboard box—for a support;

making a slit (perpendicular to the edge) in each face to sustain one plane, and parallel slits for the other plane. Of course planes not parallel can be supported in a similar way.

c. When a line is not the intersection of two planes it can be fixed in space by appearing as a line—or edge—of a fixed plane.

d. A point in space can be marked on a plane or line already fixed.

e. When it is desirable to leave the model rather open some lines can be marked by threads or rods; but generally one sees the relations of the lines more readily when they occur on planes.

(1) Prop. V. Cut a card *half* through on the dotted lines *b* and *x* and fold edges *a* and *a'* together (*a* and *a'* should be gummed together if the model is to be kept for this figure). The extra line *j* of the proof can be marked on one of the faces or left unmarked.

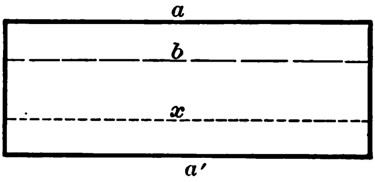

FIG. 156.

(2) Props. VI–VIII can be proved from one model.

(3) Prop. IX. Use three cards: *A, B, C.* Slide *A*

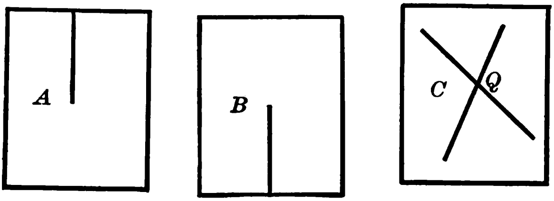

FIG. 157.

and *B* together forming two planes intersecting in the line *h*. Then push this pair of planes through *C*. *h* will pierce *C* at *Q*. *x* will be drawn in *C*.

(4) To pass a base-plane through a polyhedral angle, the level of the top edges of an open box is assumed as the plane, and the polyhedral is fixed in position in the box. By laying a straight-edge, or by stretching a string, across the top edges a point on every edge of the polyhedral may be found such that the points are coplanar.

(5) By modifying the model of note (3) a trihedral and its opposite may be formed. In the complete figure there will evidently be four pairs of inversely equal trihedrals.

II. In Book V. The figures are all closed figures, but in some cases it is better not to close the model.

(6) To make a prism, make the prismatic surface from a rectangular card, as in note (1), and fix the shape by inserting a base polygon as if for a right prism. Then determine points for a plane, oblique (if desired) to the edges, by the method of note (4).

(7) A parallelepiped surface may be conveniently cut out in one piece. The different relations of the dihedrals and face-angles should be illustrated in the parallelepipeds.

(8) One method of making a pyramid is suggested in note (4). The pyramid surface may also be cut out in one piece, the lateral triangles appearing as star-points arranged about the polygonal base as a center. It is better to make a pattern of this figure in paper and mark the cardboard from the pattern.

(9) Prop. VII.

If the entire Fig. 158 is cut out, and the lines *AB*, *BC*, *CA*, *A'B'*, are half-cut, the card will fold into a prism. Evidently this prism circumscribes a pyramid whose base is *ABC*, and whose vertex is, say, at *C'*. Suppose it is desired to make the set of three circumscribed prisms of equal altitude. Trisect AC and CC' in face F, also BC and CC' in face G, and draw the dotted lines in those faces.

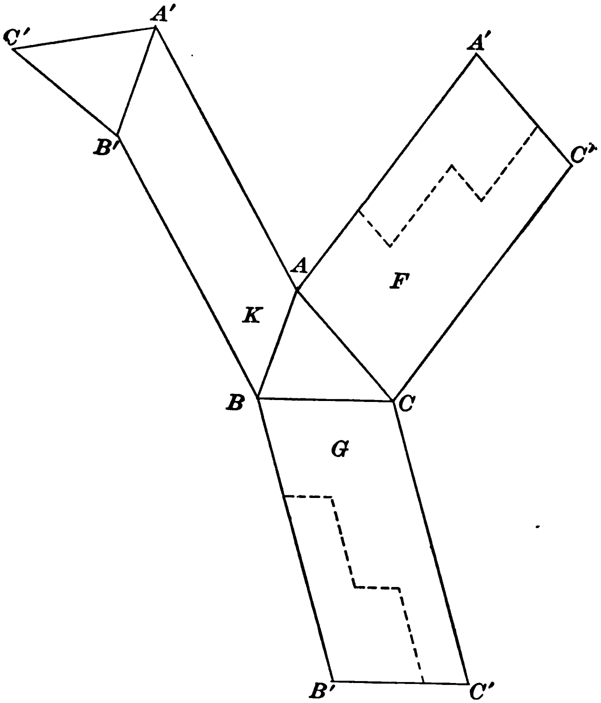

FIG. 158.

Trisect AB and BB' in face K; also in the triangle $A'B'C'$ trisect $B'C'$, and through the points draw parallels to $A'B'$ dividing the triangle into two trapezoids and a triangle. Now annex to the lowest third of face K the larger trapezoid of the triangle, then annex the next section of K, then the next trapezoid, and so on.

The complete net will have the form of Fig. 159.

The short dotted lines should be half-cut, but on the opposite face of the card from the other half-cuts.

The pupil should make his own drawing, and not attempt to trace this net; and the effect will be better if a figure rather larger than this is made.

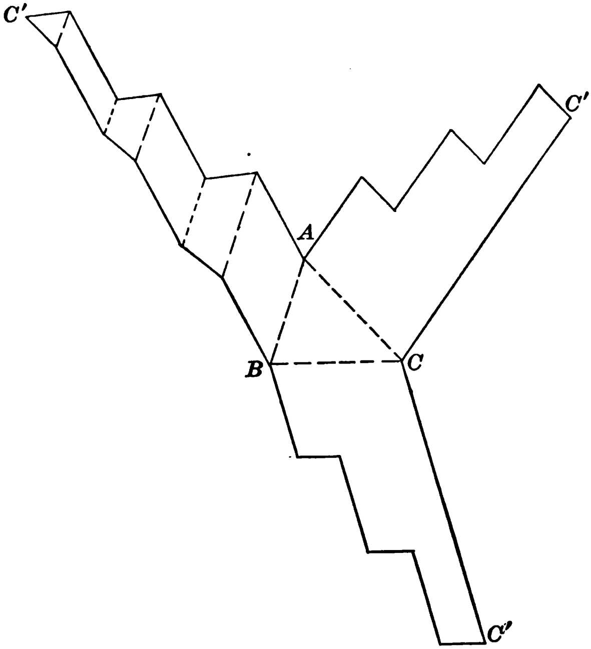

Fig. 159.

Of course the two inscribed prisms can be made from a part of the same pattern.

(10) Prop. IX. For the prism *ABCHOM* take the prism AB' used in note (9). The three tetrahedrons are easily constructed, for the four edges of each one can be measured on the outside of the prism.

(11) Truncated figures are easily constructed from suggestions already given. The theorem, sec. 484, is neatly illustrated by three parallel unequal rods standing obliquely in a base and representing the lateral edges of the prism.

Threads or elastic strings represent the other edges of the pyramids. The demonstrator can then transform a tetrahedron to its equivalent solid on the common base.

(12) A right circular cylinder is made by rolling up a rectangular sheet of paper and fitting in circular bases of the right radius. A cylindrical surface can be made oblique to its base by fitting the circular base oblique to an element. The circle must be larger than the base of the right cylinder in order to make the surface from a given sheet oblique.

(13) A right circular cone is made by rolling up a sector of a circle. The oblique circular cone is obtained as indicated in the preceding note.

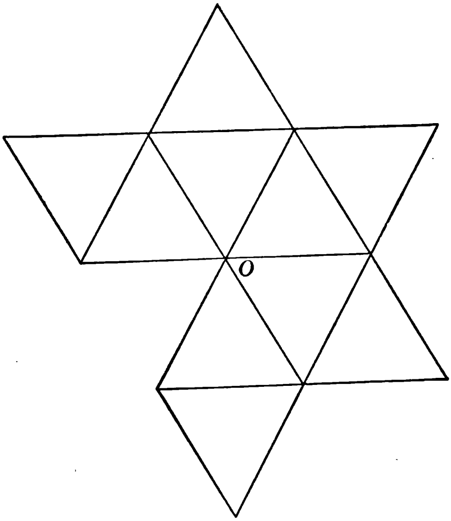

FIG. 160.

(14) The regular polyhedra.

The tetrahedron and the hexahedron have been suggested in the preceding models.

The octahedron. Construct two tetrahedral angles whose face-angles are 60° each, by drawing four equal, adjacent, equilateral triangles having a common vertex; then place the polyhedrals base to base.

The icosahedron. Construct two figures like Fig. 160. Cut the entire figure out, and cut the boundaries of the triangles half through. Bring the triangles about *O* together into a pentahedral, and bend the five star-points so that the whole figure is bent into a basket-shaped figure. The two "baskets" will fit together, the star-points of one figure fitting into the notches of the other figure.

The dodecahedron. Draw a regular pentagon and, grouped about it, five equal pentagons, each described on a side of the first as a base. Cut this out, and bend into a basket-shape, as in the icosahedron. A second basket, exactly equal to the first, will fit into it, as in the preceding solid.

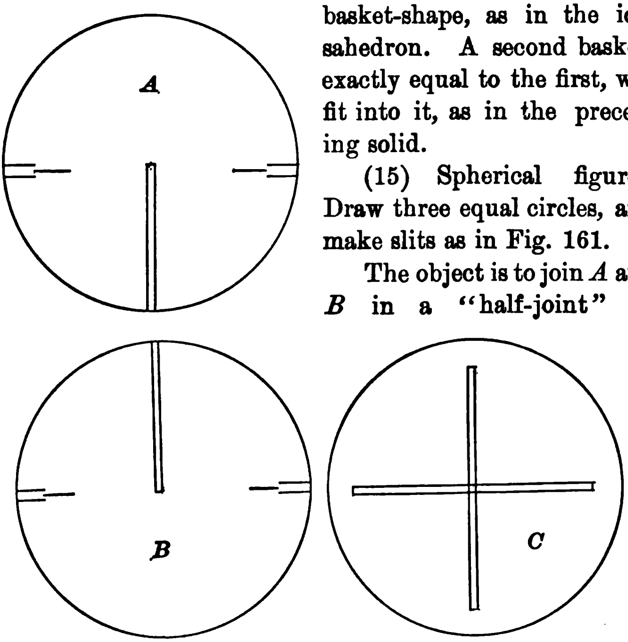

FIG. 161.

(15) Spherical figures. Draw three equal circles, and make slits as in Fig. 161.

The object is to join *A* and *B* in a "half-joint" or "lock-joint" and push them half through *C*, somewhat as in the figure for note (3). Some allowance must be made for the thickness of the card, and that is indicated by the slot in each figure. In *A* and *B* a single cut (at the right and left) is carried beyond the slot.

Cut out *A* and *B* completely, as in the figure, and *draw* the circumference of *C*, and cut the slots in it; leave some cardboard about the circle *C* for convenience in handling.

Push *A* through one slot in *C* till it comes into place, perpendicular to *C*, and locked in at the edge. Then push *B*, at right angles to both *A* and *C*, into place. The resulting figure is three great circles of a sphere, and *C* may now be trimmed down to the circumference. On such a frame spherical figures may be made by pasting narrow strips to represent large or small circles on the sphere. Spherical triangles, their symmetric triangles, polar triangles, etc., lunes, zones, segments, the whole set of spherical elements, are easily modeled on the frame.

Although the zones and lunes made from narrow strips are not strictly spherically curved, yet the general effect is excellent. The polar triangles made from strips of different colors illustrate the relations of the parts perfectly.

(16) A model for § 573 is composed of two cone surfaces —the loci of *OE* and *OL*,—and a lateral surface of a frustum of a cone—the locus of *EL*.

(17) Models for the anchor-ring and related figures are easily made by bending a piece of rubber tubing. For the ring of § 582 a rubber eraser in the shape of a triangular prism will furnish an illustration.

FOUR-PLACE LOGARITHM TABLES.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

FOUR-PLACE LOGARITHM TABLES.—*Continued.*

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

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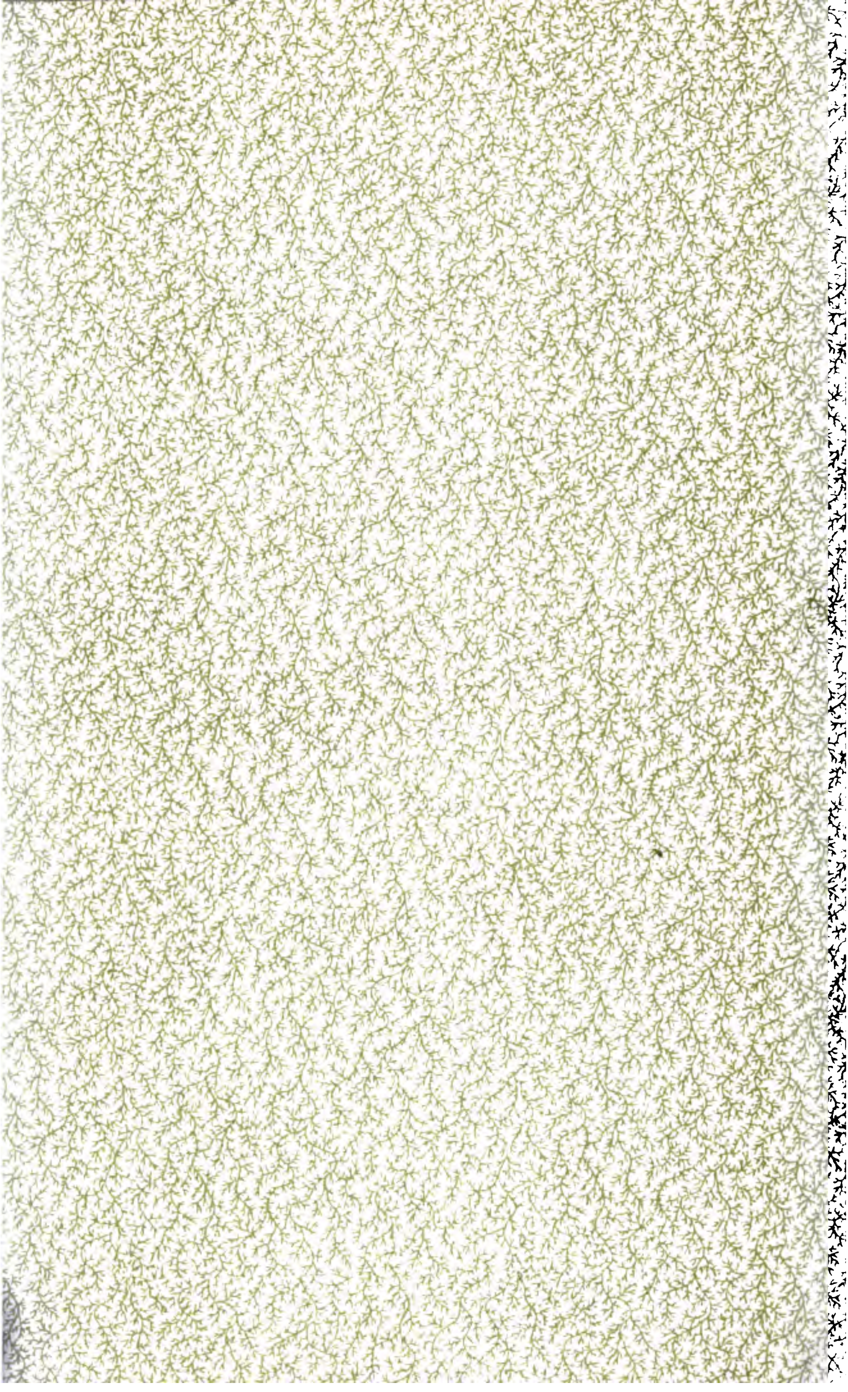

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